Applications of First Order Equations

Viscous Friction

Consider a small mass that has been dropped into a thin vertical tube of viscous fluid like oil. The mass falls, due to the force of gravity, but falls more slowly than it would in a fluid like water because the oil is thicker than water, (or, as we say, the oil is more viscous than the water). One of Newton's laws asserts that the time rate of change of the momentum of the mass is equal to the sum of the external forces acting on the mass,

$$\frac{d}{dt}(mv(t)) = F$$

where m is the mass of the object and $v(t)$ is the mass velocity as a function of time. The forces acting on the mass are the force of gravity, F_g and the friction or viscous force of the oil, F_f. Then

$$F_g = mg \qquad and \qquad F_f = -kv(t),$$

where we have assumed that the friction force is proportional to the velocity (i.e., the faster the object moves, the more the oil retards it). The constant of proportionality, k, is assumed to be constant and then the negative sign appears in the definition of F_f so that the force acts in the direction opposite to the direction of the velocity. Then we have

$$m v'(t) = mg - kv(t), \qquad v(0) = v_0, \qquad (1)$$

where v_0 denotes the initial velocity of the mass. We can rewrite the equation as

$$v'(t) = g - \frac{k}{m} v(t). \qquad (2)$$

Before solving this equation we will first carry out a qualitative analysis on the equation. We note that

if $g - \frac{k}{m} v(t) < 0,\ i.e.,\ v(t) > \frac{mg}{k} \quad then \quad v'(t) < 0\ so\ v(t)$ is decreasing

if $g - \frac{k}{m} v(t) > 0,\ i.e.,\ v(t) < \frac{mg}{k} \quad then \quad v'(t) > 0\ so\ v(t)$ is increasing

if $v(t) = \frac{mg}{k} \quad then \quad v'(t) = 0\ so\ v(t)$ is constant

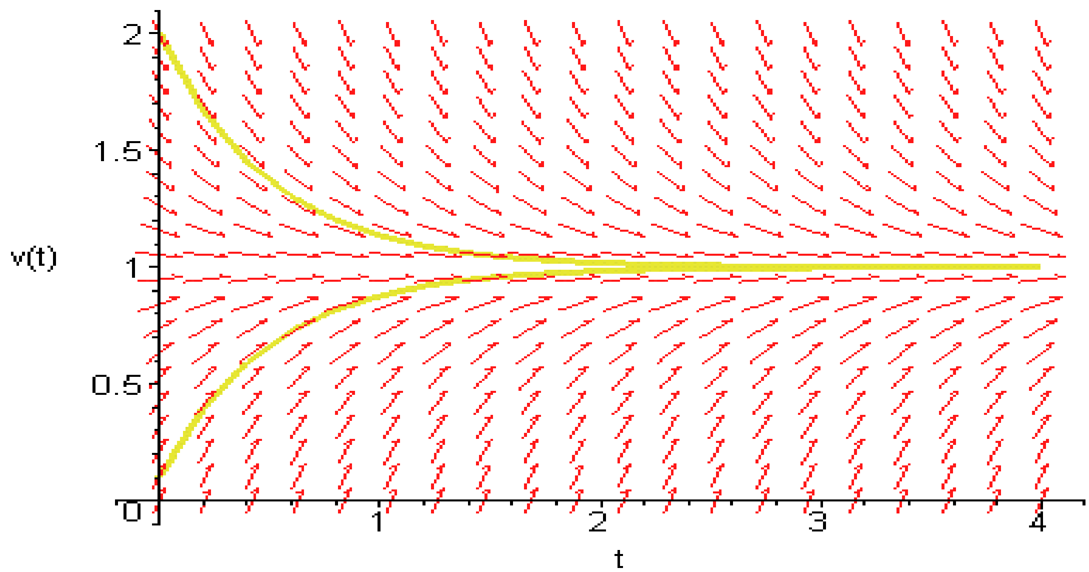
Then a solution curve that begins at a point above the value $\frac{mg}{k}$ will decrease toward this value while a solution curve starting out below this value will increase towards $\frac{mg}{k}$. In addition, we can differentiate the equation (2) to obtain

$$v''(t) = -\frac{k}{m} v'(t) = -\frac{k}{m}\left(g - \frac{k}{m} v(t)\right),$$

from which we conclude that

$$\text{if } v(t) > \frac{mg}{k} \quad then \quad v''(t) > 0 \;\; and \;\; \text{if } v(t) < \frac{mg}{k} \quad then \quad v''(t) < 0.$$

Then the solution curves above the value $\frac{mg}{k}$ all have positive curvature and those below that value have negative curvature.Then the solution curves must appear as in the following figure



The curve lying above the constant $\frac{mg}{k}$ is consistent with an object that is fired into the oil with a large initial velocity in which case the projectile gradually slows down as it continues to fall only under the effect of gravity and the viscous drag of the oil. The curve lying below the constant represents the case where the object is introduced into the tube with a small or zero initial velocity. In this case, the velocity increases toward the terminal velocity $\frac{mg}{k}$ due to gravity and friction.

In order to now solve for v(t), we note that equation (2) is equivalent to

$$\int \frac{dv}{g - \frac{k}{m}v} = \int dt,$$

from which it follows that

$$-\frac{k}{m}\ln(g - \tfrac{k}{m}v) = t + C_0$$

or

$$\ln(g - \tfrac{k}{m}v) = -\frac{mt}{k} + C_1.$$

Then

$$g - \tfrac{k}{m}v = C_2\, e^{-\frac{mt}{k}},$$

and

$$v(t) = \frac{mg}{k} - C_3 e^{-\frac{mt}{k}}.$$

The initial condition is satisfied if $\frac{mg}{k} - C_3 = v_0$ so, finally,

$$v(t) = \frac{mg}{k} - \left(\frac{mg}{k} - v_0\right)e^{-\frac{mt}{k}}.$$

This is an explicit formula for the solution curves described above. Note that $v(t) = x'(t)$ where $x(t)$ denotes the position as a function of time. Then this position function can be obtained by integrating the function $v(t)$ with respect to t.

Exercise- Find $x(t)$ and use this to solve for k if m, g are given constants and if $x(T) = d$ is known as the result of an experiment. In other words, solve for k in terms of m,g,T, and d.

A Dissolving Pill

A spherical pill is dropped into a fluid where it begins to dissolve at a rate that is proportional to the surface area exposed to the fluid. We can express this last statement in mathematical terms by writing

$$\tfrac{d}{dt}(\text{Volume}) = -K(\text{Surface Area}) \qquad (3)$$

where K denotes a positive constant of proportionality and the minus sign indicates that the volume decreases as long as the surface area is positive. For a sphere, we have

$$V = \tfrac{4}{3}\pi R(t)^3 \qquad and \qquad SA = 4\pi R(t)^2$$

where R(t) denotes the radius of the sphere (which varies with t). Then

$$\tfrac{d}{dt}(\text{Volume}) = \tfrac{d}{dt}\left(\tfrac{4}{3}\pi R(t)^3\right) = 4\pi R(t)^2 R'(t)$$

and

$$4\pi R(t)^2 R'(t) = -K\left(4\pi R(t)^2\right),$$

or

$$R'(t) = -K.$$

Then $R(t) = -Kt + C$

and $R(t) = R_0 - Kt$, where R_0 denotes the initial radius of the pill.

If the initial volume of the pill is V_0, then

$$R_0 = \left(\frac{3V_0}{4\pi}\right)^{1/3} \qquad and \qquad R(t) = \left(\frac{3V_0}{4\pi}\right)^{1/3} - Kt$$

from which it is easy to find the time to completely dissolve if K is known.

Exercise- Find out the dissolving time for a cubical pill, assuming that the pill maintains its cubical shape as it dissolves. Which dissolves in less time, the cubical pill or the spherical pill?

Water Heating Strategy

Suppose a cylindrical water heater contains a heating element that electrically heats the water in the tank (which we suppose to be well insulated from the outside). We have already seen this model, which we can write in the form

$$\frac{d}{dt}T(t) = k(H_0 - T(t)), \qquad T(0) = T_0, \qquad (4)$$

where H_0 denotes the temperature to which the water will be eventually heated if the heating element is left on constantly, T_0 is the temperature at which the water enters the tank, and k is a positive constant that reflects the proportionality between the time rate of change in the water temperature and the difference between H_0 *and* $T(t)$. We know that we can solve this equation by writing

$$\int \frac{dT}{H_0 - T} = k\int dt$$

which leads to $\quad -\ln(H_0 - T(t)) = kt + C_0$

and

$$T(t) = H_0 + C_1 e^{-kt}.$$

It follows from the initial condition that

$$T(t) = H_0 + (T_0 - H_0)e^{-kt}. \qquad (5)$$

This solution predicts that the temperature will increase from T_0 toward the limiting value H_0 at an exponential rate that is determined by the constant k. Suppose, however, that when the water temperature reaches some value $T_1 < H_0$ at a time t_1, we turn off the water heater for 3 hours. Then the water will gradually cool, as a result of heat loss to the exterior. If the reduced water temperature at the end of 3 hours is still within acceptable range, we could then turn the heater back on until the water heats to the original temperature T_1, at which point we could then turn the heater off again for 3 hours and repeat the whole cycle over and over. This might be a more economical strategy for heating water than to leave the heater on at all times. It only remains to determine if the water temperature after 3 hours is within an acceptable range.

In order to determine the water temperature at the end of the 3 hours, we have to solve the following initial value problem,

$$\frac{d}{dt}T(t) = K(S - T(t)), \qquad T(t_1) = T_1, \qquad (6)$$

where S denotes the temperature of the outside (we suppose $T(t) > S$) and K is a positive constant expressing the proportionality between the rate of change of T(t) and the difference $S - T(t)$. Comparing equation (6) with equation (4), it then follows from (5) that this solution is given by

$$T(t) = S + (T_1 - S)e^{-K(t-t_1)} \qquad (7)$$

which predicts that the temperature of the water will begin decreasing at $t = t_1$ toward the limiting value of S. Three hours after time $t = t_1$, the temperature will be equal to

$$T(3) = S + (T_1 - S)e^{-3K} = T_2 < T_1.$$

If this is an acceptable (not too cold) temperature, we can turn the heater on and compute the time needed to return the temperature to the level $T = T_1$. We then program the heater to be on for this number of hours and off for three hours and to continue this on/off sequence indefinitely.

Another strategy would be to place temperature sensors in the water heater which would turn the heater off when the water temperature reaches T_1 and would keep the heater off until the water temperature decreases to T_2. Would the amount of time the heater is on in this strategy be the same as the time the heater is on using the previous strategy?

Absorption of Medications

When you take a pill to obtain medication, the pill first goes into your stomach and the medication passes into your GI tract. From there the medication is absorbed into your bloodstream and circulated through your body before being eliminated from the blood by the kidneys and other organs. If we let $x(t)$ denote the amount of medication in your GI tract at time t, then we can model the movement of the medication out of the GI tract with the equation

$$x'(t) = -k_1 x(t), \qquad x(0) = A. \qquad (8)$$

This is the assertion that after taking the pill, an amount A of medication is in the GI tract and it decreases at a rate proportional to the amount currently present in the GI tract. If the amount of medication in the bloodstream at time t is denoted by $y(t)$, then

$$y'(t) = k_1 x(t) - k_2 y(t), \qquad y(0) = 0, \qquad (9)$$

expresses the fact that medication is coming into the bloodstream at exactly the rate it is leaving the GI tract and it is leaving the bloodstream at some rate expressed by the proportionality constant k_2. Also, we are assuming that there is no medication in the bloodstream initially. Now this is two equations for the two unknown functions $x(t)$ *and* $y(t)$, but the first equation can be solved independently and the solution substituted into the equation (9).

The solution of (8) is easily found to be

$$x(t) = A e^{-k_1 t},$$

and then $$y'(t) + k_2 y(t) = k_1 x(t) = k_1 A e^{-k_1 t}.$$

We will find a particular solution for the y-equation by the method of undetermined coefficients. We guess that $y_p(t) = a e^{-k_1 t}$ and then

$$y_p'(t) + k_2 y_p(t) = -k_1 a e^{-k_1 t} + k_2 a e^{-k_1 t} = k_1 A e^{-k_1 t}.$$

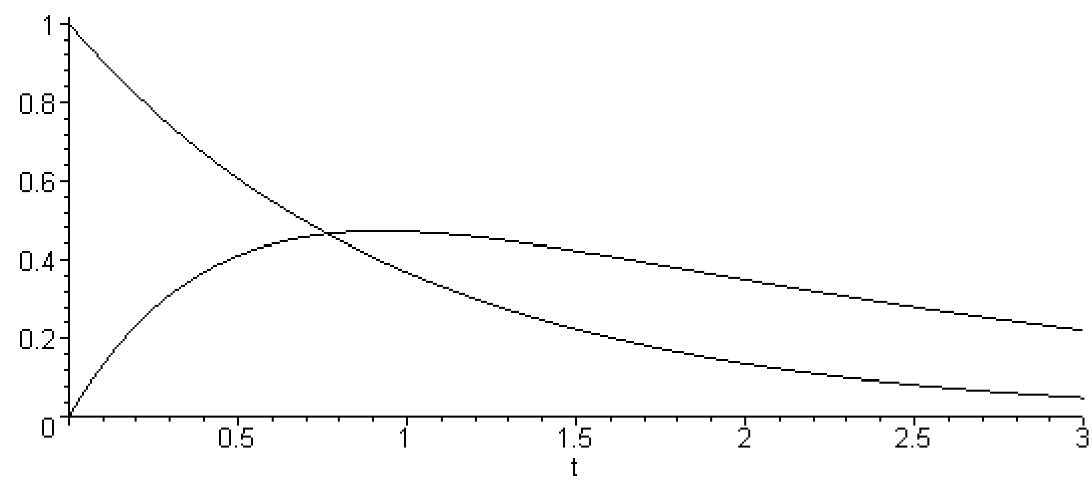
This leads to $$a = \frac{k_1 A}{k_2 - k_1} \qquad \text{and} \qquad y_p(t) = \frac{k_1 A}{k_2 - k_1} e^{-k_1 t}.$$

Then
$$y(t) = C e^{-k_2 t} + \frac{k_1 A}{k_2 - k_1} e^{-k_1 t}$$

and, using the initial condition to evaluate C, we get

$$y(t) = \frac{k_1 A}{k_2 - k_1}[\, e^{-k_1 t} - e^{-k_2 t}]$$

Plotting x(t) and y(t) versus t for some representative values of the constants gives the following figure



x(t) and y(t) versus t

If we wish to consider a model that represents taking a continuously acting pill, (a pill that releases medication continuously so as to maintain a constant level of medication in the GI tract for a sustained period of time), we might modify the previous model $(8), (9)$ to read

$$x'(t) = X_0 - k_1 x(t), \qquad x(0) = 0.$$

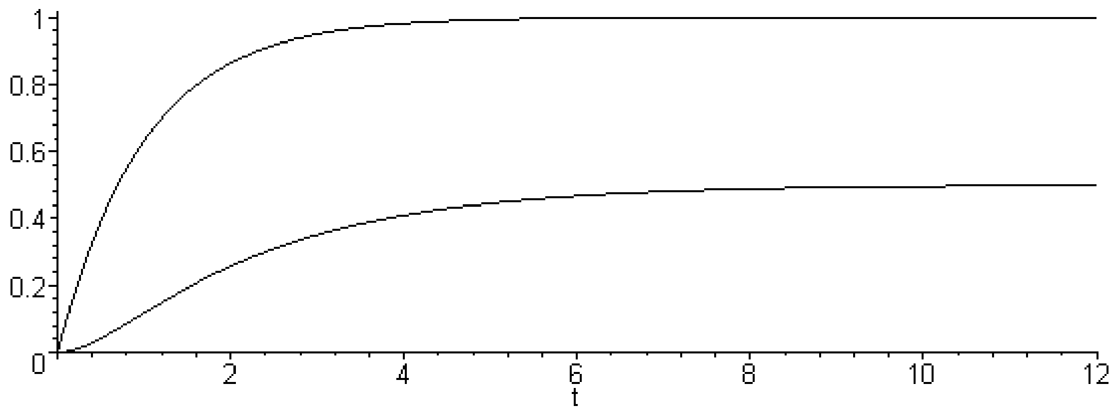
$$y'(t) = k_1 x(t) - k_2 y(t), \qquad y(0) = 0.$$

In this case, we find

$$x(t) = \frac{X_0}{k_1}(1 - e^{-k_1 t})$$

$$y(t) = \frac{1}{k_2}\left[1 + \frac{1}{k_2 - k_1}(k_1 e^{-k_2 t} - k_2 e^{-k_1 t}) \right]$$

Plotting these solutions gives,



upper curve=x(t) and lower curve=y(t)

Clearly this produces a longer constant level of medication in the bloodstream. Of course, eventually, the level of medication in the GI tract will go to zero and the level in the bloodstream will then also decrease to zero. A model which could describe the periodic taking of a sequence of time release pills would look like

$$x'(t) = X_0(t) - k_1 x(t), \qquad x(0) = 0.$$

$$y'(t) = k_1 x(t) - k_2 y(t), \qquad y(0) = 0,$$

where $X_0(t)$ denotes a piecewise constant function that alternates between a positive value and zero. We will discuss an effective way to solve equations involving such terms when we discuss the Laplace transform.