Second Order Linear Equations

Linear Equations

The most general linear ordinary differential equation of order two has the form,

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t). \qquad (1)$$

We call this a ***linear*** equation because the unknown function $y(t)$ and its derivatives appear in the equation in a ***linear*** way. That is, there are no products of y and its derivatives, no powers higher than one, no nonlinear functions with y or its derivatives as an argument. But there is a more satisfactor way to define what we mean by a linear equation. First, define a "function" into which we substitute functions of t rather than numbers,

$$L[\bullet] = a(t)\,\tfrac{d^2}{dt^2}[\bullet] + b(t)\,\tfrac{d}{dt}[\bullet] + c(t)[\bullet]. \qquad (2)$$

i.e.,

$$L[y_1(t)] = a(t)\,\tfrac{d^2}{dt^2}[y_1(t)] + b(t)\,\tfrac{d}{dt}[y_1(t)] + c(t)[y_1(t)]$$
$$L[y_2(t)] = a(t)\,\tfrac{d^2}{dt^2}[y_2(t)] + b(t)\,\tfrac{d}{dt}[y_2(t)] + c(t)[y_2(t)].$$

We refer to $L[\bullet]$ as an operator. Then it is easy to see that

$$L[y_1(t)+y_2(t)] = a(t)\,\tfrac{d^2}{dt^2}[y_1(t)+y_2(t)] + b(t)\,\tfrac{d}{dt}[y_1(t)+y_2(t)] + c(t)[y_1(t)+y_2(t)]$$

$$= a(t)\,\tfrac{d^2}{dt^2}[y_1(t)] + a(t)\,\tfrac{d^2}{dt^2}[y_2(t)] + b(t)\,\tfrac{d}{dt}[y_1(t)] + b(t)\,\tfrac{d}{dt}[y_2(t)]$$

$$+ c(t)[y_1(t)] + c(t)[y_2(t)]$$

$$= L[y_1(t)] + L[y_2(t)].$$

Similarly, for any constant k,

$$L[ky(t)] = a(t)\,\tfrac{d^2}{dt^2}[ky(t)] + b(t)\,\tfrac{d}{dt}[ky(t)] + c(t)[ky(t)]$$

$$= ka(t)\,\tfrac{d^2}{dt^2}[y(t)] + kb(t)\,\tfrac{d}{dt}[y(t)] + kc(t)[y(t)]$$

$$= kL[y(t)].$$

These two observations can be combined in the following single assertion,

$$L[C_1 y_1(t) + C_2 y_2(t)] = C_1 L[y_1(t)] + C_2 L[y_2(t)], \qquad (3)$$

which holds for all constants C_1, C_2 *and all functions* $y_1(t), y_2(t)$. Any operator L having property (3) is said to be a **linear operator**. A differential equation, like (1), for which the associated operator, (2), is linear is said to be a linear differential equation. The word linear derives from the fact that a real valued function $f(x)$ with the property that $f(C_1x_1 + C_2x_2) = C_1f(x_1) + C_2f(x_2)$, has a straight line for its graph of f(x) versus x. The new definition of linear differential equation is equivalent to the previous definition.

Note that the linearity condition (3) implies that if $y_1(t), y_2(t)$ are both solutions of the homogeneous equation $L[y(t)] = 0$, then the combination $C_1y_1(t) + C_2y_2(t)$ is also a solution of the same equation for all choices of the constants C_1, C_2. This is known as the **principle of superposition** and it holds for all linear homogeneous equations.

Equations With Constant Coefficients

An equation of the form (1) is still more difficult to solve than we are prepared to handle in this course. We will consider the simpler situation in which the coefficients in the equation are all constants. This is an equation of the form,

$$ay''(t) + by'(t) + cy(t) = f(t). \qquad (4)$$

and we will begin by considering the even simpler situation where the equation is homogeneous,

$$ay''(t) + by'(t) + cy(t) = 0. \qquad (5)$$

To solve this equation we note that if $y(t) = e^{rt}$ is substituted into (5) we get

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt} = 0,$$

and this is satisfied if and only if r solves $ar^2 + br + c = 0$. If the roots of this quadratic equation are denoted by r_1 *and* r_2 then e^{r_1t} *and* e^{r_2t} are both solutions of the equation (5). By the principle of superposition then $C_1e^{r_1t} + C_2e^{r_2t}$ is also a solution of (5) for all choices of the constants C_1, C_2. Then the problem of finding solutions for (5) is reduced to the problem of finding the roots of the **auxilliary equation**

$$ar^2 + br + c = 0 \qquad (6)$$

Examples

i) Consider the equation $\quad y''(t) + 2y'(t) - 3y(t) = 0.$

The auxilliary equation is $\quad r^2 + 2r - 3 = 0$

with roots $\quad r = 1, -3.$

Then $\quad y_1(t) = e^t \;\; and \;\; y_2(t) = e^{-3t}$

are both solutions for the differential equation. The function $y(t) = C_1 e^t + C_2 e^{-3t}$ is also a solution for all choices of C_1, C_2.

ii) Consider the equation $$y''(t) + 36y(t) = 0.$$

The auxilliary equation in this case is $$r^2 + 36 = 0$$

with roots $$r = 6i, \ -6i \quad where\ i = \sqrt{-1}$$

Then $$y_1(t) = e^{i6t} \quad and \quad y_2(t) = e^{-i6t}$$

are both solutions for the differential equation. However, these solutions are both complex valued functions and we had been expecting a real valued solution. If we recall the definitions of the sine and cosine functions

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad and \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

then we see that we can form new solutions to the equation as follows

$$y_3(t) = \frac{y_1(t) + y_2(t)}{2} = \cos(6t) \qquad and \qquad y_4(t) = \frac{y_1(t) - y_2(t)}{2i} = \sin(6t).$$

The functions $y_3(t)\ and\ y_4(t)$ are two new solutions to the differential equation and $y(t) = C_1 \cos(6t) + C_2 \sin(6t)$ is also a solution for all choices of C_1, C_2.

iii) Consider the equation $$y''(t) + 2y'(t) + 36y(t) = 0.$$

The auxilliary equation in this case is $$r^2 + 2r + 36 = 0,$$

with roots $$r = -1 \pm i\sqrt{35}.$$

Then $$y_1(t) = e^{(-1+i\sqrt{35})t} = e^{-t} e^{\left(i\sqrt{35}\right)t}$$
and $$y_2(t) = e^{(-1-i\sqrt{35})t} = e^{-t} e^{\left(-i\sqrt{35}\right)t}$$

are both solutions for the differential equation. In this case, forming the combinations

$$y_3(t) = \frac{y_1(t) + y_2(t)}{2} \qquad and \qquad y_4(t) = \frac{y_1(t) - y_2(t)}{2i},$$

leads to
$$y_3(t) = e^{-t} \cos\left(t\sqrt{35}\right) \qquad and \qquad y_4(t) = e^{-t} \sin\left(t\sqrt{35}\right).$$

Evidently, when the roots of the auxilliary equation are the complex conjugate pair, $r = \alpha \pm i\beta$, then the corresponding real valued solutions of the differential equation are

$$y_3(t) = e^{\alpha t}\cos(\beta t) \qquad \textit{and} \qquad y_4(t) = e^{\alpha t}\sin(\beta t),$$

and $y(t) = e^{\alpha t}[C_1\cos(\beta t) + C_2\sin(\beta t)]$ is also a solution for all choices of C_1, C_2.

iv) Finally, consider the equation $\qquad y''(t) + 2y'(t) + y(t) = 0.$

The auxilliary equation in this case is $\qquad r^2 + 2r + 1 = 0,$

with repeated root $\qquad r = 1, 1.$

Then one solution of the equation is the function $y(t) = e^t$, but we expect a second solution. To see what it should be, we note that substituting the guess e^{rt} into the differential equation (5) led to

$$L[e^{rt}] = P(r)\,e^{rt} = 0, \quad \textit{where} \quad P(r) = ar^2 + br + c.$$

Then we observed that e^{rt} solves the differential equation if r is a root of the auxilliary equation. When the auxilliary equation is

$$P(r) = r^2 + 2r + 1 = 0,$$

we have not only $P(1) = 0$, we have $P'(1) = 0$ as well. That is, when a quadratic equation has a double root at $r = r_0$ then we have

$$P(r_0) = P'(r_0) = 0.$$

Then notice that since $\qquad \frac{d}{dr}L[e^{rt}] = \frac{d}{dr}(P(r)\,e^{rt})$

and $\qquad \frac{d}{dr}L[e^{rt}] = L\left[\frac{d}{dr}e^{rt}\right] = L[t\,e^{rt}],$

$$\frac{d}{dr}(P(r)\,e^{rt}) = P'(r)e^{rt} + P(r)\,re^{rt},$$

we have $\qquad L[t\,e^{r_0 t}] = P'(r_0)e^{r_0 t} + P(r_0)\,r_0 e^{r_0 t} = 0.$

That is, when the quadratic auxilliary equation has a double root at $r = r_0$, then two solutions of the differential equation are the functions,

$$y_1(t) = e^{r_0 t} \qquad \textit{and} \qquad y_2(t) = t\,e^{r_0 t}$$

and then the function $y(t) = C_1 e^{r_0 t} + C_2\, t e^{r_0 t}$ is also a solution for all choices of C_1, C_2.■

In each of the previous examples, the equation had two solutions, and in each case these two solutions were what we call "linearly independent".

Definition Functions $y_1(t), y_2(t), \ldots, y_N(t)$, all of which are defined on an interval, I, are said to be *linearly independent* if the following statements are equivalent:

i) $C_1 y_1(t) + C_2 y_2(t) + \ldots, + C_N y_N(t) = 0 \quad \textit{for all } t \textit{ in } I$

ii) $C_1 = C_2 = \ldots = C_N = 0$

It is obvious that ii) always implies i) but the converse is only true if the functions are linearly independent on I. Essentially, a set of functions is linearly independent if none of the functions can be expressed as linear combinations of the remaining functions. We will prove later that every linear second order homogeneous differential equation has exactly two linearly independent solutions. Note that in examples ii) and iii) we found four solutions for the homogeneous equations there but in both cases, there are only two independent solutions. The remaining two solutions can be expressed as linear combinations of the other two solutions. In the four examples above, we have found two linearly independent real valued solutions for the homogeneous differential equation in the example. We can summarize what we found in these examples in the following table.

roots of the aux eqn	L.I. solns of homog ODE	
distinct real *roots* r_1, r_2	$y_1(t) = e^{r_1 t}$,	$y_2(t) = e^{r_2 t}$
imaginary pair $r = \pm i\omega$	$y_1(t) = \cos(\omega t)$,	$y_2(t) = \sin(\omega t)$
conjugate pair $r = \alpha \pm i\omega$	$y_1(t) = e^{\alpha t}\cos(\omega t)$,	$y_2(t) = e^{\alpha t}\sin(\omega t)$
double real *root* r_1, r_1	$y_1(t) = e^{r_1 t}$,	$y_2(t) = t e^{r_1 t}$

If $y_1(t)$ *and* $y_2(t)$ are linearly independent solutions of the homogeneous equation (5), then the **general solution** of (5) is given by $y(t) = C_1 y_1(t) + C_2\, y_2(t)$. The general solution contains two arbitrary constants and we say, therefore, that the general solution of a second order homogeneous ODE is a 2-parameter family of solutions. This allows us to solve initial value problems of the form

$$ay''(t) + by'(t) + cy(t) = 0, \qquad y(t_0) = A_0, \quad y'(t_0) = A_1.$$

Once the general solution $y(t) = C_1 y_1(t) + C_2\, y_2(t)$ has been found, we simply write

$$\begin{aligned} y(t_0) &= C_1 y_1(t_0) + C_2\, y_2(t_0) = A_0 \\ y'(t_0) &= C_1 y_1'(t_0) + C_2\, y_2'(t_0) = A_1, \end{aligned}$$

and we solve this set of two equations for the two unknowns C_1 *and* C_2. The resulting function with these two values for C_1 *and* C_2 is then the unique solution to the initial value problem.

Example $y''(t) + 2y'(t) - 3y(t) = 0, \quad y(0) = 1, \ \ y'(0) = -3.$

The general solution for the differential equation was found in a previous example to be,

$$y(t) = C_1e^t + C_2e^{-3t}.$$

Then the initial conditions imply

$$\begin{aligned} y(0) &= C_1e^0 + C_2e^0 = C_1 + C_2 = 1 \\ y'(0) &= C_1e^0 - 3C_2e^0 = C_1 - 3C_2 = -3, \end{aligned}$$

and

$$C_1 = 0, \quad C_2 = 1.$$

Then $y(t) = e^{-3t}$ is the unique solution of the initial value problem.