An Approximate Integral Scheme of Calculating the Transitional Boundary Layer in Two-Dimensional Incompressible Flow

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ABSTRACT

Integral calculations of two-dimensional, incompressible, thermal, transitional boundary layers have been performed. To precede these approximate calculations, mathematical model was developed in order to enable prediction of the main boundary layer integral parameters. The model was proposed to calculate the characteristics of the boundary layers under the effect of local heat transfer and moderate free-stream turbulence levels by enhancing established integral techniques in conjunction with intermittency weighted model of the transitional boundary layer. Empirical relationships for the prediction of the start and end of transition, as well as the development of the boundary layer during the transition region were based on results of experimental investigations. Since the heat transfer coefficient between external flow and surface is extremely influenced by the level of turbulence in the flow, it is also found to be very sensitive to the solid surface temperature and thereby an adequate solution of the thermal boundary layer is required. To satisfy these conditions, the mathematical model included both dynamic and thermal boundary layer equations in integral form. To support the results validation, a numerical investigation utilized Menter el. al [7] model in ANSYS-CFX tool has been represented beside the experimental results.

Keywords: Transition onset; Intermittency; Integral scheme; Transport equation model; Boundary layer; Thermal boundary layer; Turbulence intensity

NOTATION

\( \frac{\tau}{(1/2 \cdot \rho \cdot U^2)} \) skin friction coefficient; \( c_p \) specific heat, \( J/(kg \cdot K) \)
\( \dot{\rho} \) mass flux, \( \rho \cdot U, kg/(s \cdot m^2) \)
\( H \) shape factor, \( \delta^* / \theta \)
\( h \) convective heat transfer coefficient, \( W/(m^2 \cdot K) \)
\( k \) thermal conductivity, \( W/(m \cdot K) \)
\( L \) total length of the plate, \( m \)
\( Nu \) Nusselt number, \( (h \cdot x)/k = St \cdot Re_x \cdot Pr \)
\( Pr \) Prandtl number, \( \mu \cdot cp/k \)
\( q \) heat flux, \( J/kg \)
\( Re_L \) Reynolds number based on length of transition region, \( L \cdot U_{\infty}/\nu \)
\( Re_x \) Reynolds number based on streamwise distance on plate surface, \( x \cdot U_{\infty}/\nu \)
\( Re_{\Delta_2} \) Reynolds number based on enthalpy thickness, \( \Delta_2 \cdot U_{\infty}/\nu \)
\( Re_\theta \) Reynolds number based on boundary layer thickness, \( \delta \cdot U_{\infty}/\nu \)
\( St \) Stanton number, \( h/(\rho \cdot U_{\infty} \cdot c_p) \)
\( \Delta T \) temperature difference between local flow and external flow, \( K \)
\( \Delta_2 \) boundary layer enthalpy thickness, \( m \)
\( \Delta_3 \) boundary layer conduction thickness, \( m \)
\( \delta \) boundary layer thickness, \( m \)
\( \delta^* \) displacement thickness, \( m \)
\( \eta \) non-dimensional length during transition
\( \theta \) momentum thickness, \( m \)
\( \kappa \) adiabatic exponent, specific heat ratio
\( \lambda \) pressure gradient parameter, \( \frac{\theta^2 \cdot du_{\infty}}{\nu \cdot dx} \)
\( \mu \) dynamic viscosity, \( Pa \cdot s \)
\( \nu \) kinematic viscosity, \( m^2/s \)
\( \rho \) density, \( kg/m^3 \)
\( \sigma \) transition region extension standard deviation, \( m \)
\( \tau \) shear stress, \( Pa \)
\( \gamma \) intermittency factor

Subscripts

\( E \) end of transition
\( e \) external edge of boundary layer
\( lam \) laminar regime
\( m \) transition region mean position
\( \lambda \) start of transition
\( tr \) transition onset
\( turb \) turbulent regime
\( w \) flat plate wall
\( \infty \) local freestream
\( o \) start of thermal boundary layer

During transition region, the ratio of integral parameters are denoted by \( ' \)
1. INTRODUCTION
The subject of laminar-turbulent transition is of considerable practical interest and has a wide range of engineering applications due to the fact that transition controls the evolution of important aerodynamic quantities such as drag or heat transfer. Transition in boundary layer flows in turbomachines and aerospace devices is known to be affected by various parameters, such as freestream turbulence, pressure gradient and separation, Reynolds number, Mach number, turbulent length scale, wall roughness, streamline curvature and heat transfer. Due to this variety of parameters, there is no mathematical model exist that can predict the onset and length of the transition region. In addition to the influence of these parameters upon transition origination, the poor understanding of the fundamental mechanisms which lead initially small disturbances to transition may also caused this lack. At present, there are three main concepts used to model transition in industry. The first approach is based on the stability theory where the successful technique is so-called $e^N$ method. This method is based on the local linear stability theory and the parallel flow assumption in order to calculate the growth of the disturbance amplitude from the boundary layer neutral point to the transition location. A shortcoming of this technique indicates that it is not compatible with the current CFD methods because the typical industrial Navier-Stokes solutions are not accurate enough to evaluate the stability equation. Moreover since it is based on the linear stability theory, it cannot predict the transition due to non-linear effects such as high freestream turbulence or surface roughness. The second approach uses the conventional turbulence models such as the two-equation turbulence model of Lauder and Sharma [1]. The disadvantages of this solution that first, ignores the transition physics and the importance of the transition zone completely and secondly, it is fabricated especially to deals with flows where the transitional region covers a large portion of the flow field. The main concept of the construction of these models that, the calibration of the damping functions in these models is based on reproducing the viscous sublayer behavior, not on predicting transition from laminar to turbulent flow. The last approach is usage of the concept of intermittency to blend the flow from laminar to turbulent regions. The development of intermittency in this technique is based on the observations from the experimental work. Due to these observations, empirical relationship can be established to correlate the onset location and growth rate of the transition. To achieve this task, most correlations usually relate the important affected parameters in the physical domain of study, such as free-stream turbulence level, $Tu$, and pressure gradient to the transition momentum thickness Reynolds number. Well-known correlations in literature are that of Mayle [2] and Abu-Ghannam and Shaw [3]. This technique is quite often used for the steady boundary layer on a flat plate. The empirical correlation of the transition region can be used within differential methods such as Forest [4], McDonald and Fish [5], Arnal et al. [6], Menter et al. [7] and Cebeci and Cousteix [8] and solved numerically to predict the development of the transitional boundary layer or can be included with existed approximate integral methods of laminar and turbulent boundary layers such as in Abu-Ghannam and Shaw [3], Fraser, C. J. and Milne, J. S. [9], Davenport, Schetz and Wang [10], Chris Kirney [11] and Martin Hepperle [12] and thus the entire development of boundary layer can be predicted.

The main aim of the present paper is to utilize well-structured integral methods of laminar and turbulent flows with empirical correlation of transition region to build up approximate integral scheme with higher capability of calculating the momentum and thermal boundary layers subjected to free-stream turbulence intensity within the transitional region in two dimensional incompressible flows.

2 PHYSICAL MODEL OF TRANSITION
Transition process from laminar to turbulent flow is demonstrated as a result of a sequence of complicated phenomena which are influenced by many factors. Part of these factors is due to the environmental flow conditions, whereas the remaining factors are emanating due to generated excitation respect to flow exhibition around artificial constructions. The level of influenced factors upon transition process is mainly appearing in the way of how the transition to turbulence exists.

The structural development in the natural transition region of boundary layer follows a certain sequence process. According to the theory of stability, the first step in the transition process is the presence of self-excited disturbances in the laminar boundary layer. The growth of these small disturbances so called TS-waves follows the exponential law in the first propagation, thus can be describes by linear stability theory. Further downstream, when the perturbations reach certain amplitude, their propagation starts to deviate from that predicted by linear growth. The initially two-dimensional Tollmien-Schlichting waves with respect to some experiments investigations are distorted into a series of "Peaks" and "valleys", known as $A - \text{structures}$. The formation of $A - \text{structures}$ downstream is due to superimposed of the three-dimensional disturbances caused by secondary instabilities.

Further downstream, three-dimensional and nonlinear effects are increased. Due to the nonlinear development of the disturbances the peak-valley structures are stretched and form horseshoe vortices. These $A - \text{vortices}$ decay downstream into small and small vortices which finally replaced by turbulent "spots". The onset of transition can be defined in the exact location of streamwise where the first spots are presented. At this location the velocity profile is reshaped from that profile of the laminar plate boundary layer solved by Blasius to the profile of the fully turbulent plate boundary layer. This is revealed in strong decrease in the shape factor H, while a great increase in the friction drag is observed. Moreover, a great increase in the boundary-layer thickness occurs [13]. Continuous developing of the turbulent spots initiates the
transition to fully turbulent boundary-layer flow as the last stage of transition process.

In transition region, the isolated spots within which a fully developed turbulent flow exists appear successively in a random fashion in time and space and grow as they are washed downstream. Thus the flow at any point in the region becomes turbulent during those periods of time during which a spot moves over it and is laminar for the remainder.

Although it is well-known that the transition is random phenomenon, it may be possible to determine the fraction of the total time in which the flow is turbulent as an average taken over an appropriate interval of time. In fact this fraction, which is called intermittency factor \( \gamma \), has been determined by Schubauer and Klebanoff [14] and Dhawan and Narasimha [15] from their experimental data record. In the same manner it can be reasonable to assume the existence of a sort of probability function specifying the rate of spot formation per unit area.

The intermittency factor which depends on the rate of spot formation and of its subsequent growth [16] is suggested by Abu-Ghannam and Shaw [3] by the following relationship

\[
\gamma = 1 - \exp(-5\eta^2)
\]  

(1)

It was deduced from the amplitude density function \( p(u) \) of the signal measured directly by the amplitude probability analysis (APA) system.

To evaluate the parameters behavior within the laminar-turbulent transition region, some authors [17] admitted a linear combination of turbulent and laminar results weighted by an exponential probability function (intermittency) which was developed based on the following relationship

\[
\Gamma = \frac{1}{2} \left[ 1 - \exp(-5\eta^2) \right]
\]  

In this case, the relationship of Reynolds et al. [18] that defines the transition region statistically by a mean position \( x_m \) and a standard deviation length \( \sigma \) can be used to defined the integral parameters \( C_{ftr} \) and \( S_{ttr} \) within the region by

\[
C_{ftr}(x) = \begin{cases} 
C_{lam}(Re_x), & x < x_m - 2\sigma \\
1 - \gamma(x)C_{lam}(Re_x) + \gamma(x)C_{turb}(Re_x), & x \geq x_m - 2\sigma 
\end{cases}
\]  

(2)

where,

\[
\gamma(Re_x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(Re_x - Re_{2m})^2}{2Re_{2m}}\right) d(Re_x)
\]  

(3)

In the present program, only the proposed relations of [3] have been used to evaluate the development of the integral parameters.

3 MATHEMATICAL MODEL OF TRANSITION
The mathematical model developed in this work requires:

1) The solution of velocity field over the flat plate with sharp-leading edge.

2) The calculation of the momentum and energy boundary layers equations in integral form.

In most numerical codes used to solve the boundary layers around airfoils, the program first compute the potential flow around a 2-D airfoil using the vortex panel method and then uses the results of this method to compute the boundary layers on the two surfaces of the airfoil [11].

In the present program you have to specify directly the freestream velocity and the turbulence level in the flow as input parameters. Since the scheme is limited to flows with constant thermophysical properties and isothermal surface temperature, the rest of input parameters are standard and defined inside the program. With this standard input data, the proposed mathematical model is able to predict operational parameters along the flat plate through different types of boundary layers. These parameters such as skin-friction coefficient, momentum thickness and shape factor beside the thermal boundary layers integral parameters such as convective heat transfer coefficient and thermal boundary layer thickness. The physical domain of the mathematical model is presented in Fig. 1.

![Figure 1: Physical domain of the mathematical model](image)

4 CALCULATIONS OF BOUNDARY LAYERS INTEGRAL PARAMETERS
The evaluation of the integral parameters of momentum boundary layer and thermal boundary layer is performed by solving the dynamic and thermal boundary layers equations in integral form at laminar and turbulent regimes. For laminar-turbulent transition regime, Abu-Ghannam and Shaw [3] empirical relationship is adopted.

In the present program, the governing equations (4) and (6) (momentum and energy boundary layer equations respectively) are simplified considering a steady-state, one-dimensional flow over a smooth, and impermeable flat plate surface. The program also is limited to incompressible flows with constant thermophysical properties and a constant wall temperature. It is apple to treat cases with sharp leading edge, planar geometry and arbitrary free-stream turbulence levels and velocities.

The boundary layer momentum-integral equation by Karman [19]

\[
f = \frac{df}{dx} + (2 + H) \frac{\partial}{\partial x} \frac{d\theta}{dx}
\]  

(4)

where, \( H \) is the shape factor defined by

\[
H = \frac{\delta^*}{\theta}
\]  

(5)

The thermal-energy integral equation by Frankl [20]

\[
q_w = \frac{d}{dx} \left[ \int_0^\infty \rho c_p u (T - T_e) dy \right]
\]  

(6)
4.1 CALCULATION OF MOMENTUM-BOUNDARY LAYERS PARAMETERS

The laminar boundary layer
Thwaites-Walz integral method [21] & [22] is used to calculate the laminar boundary layer parameters starting from the sharp-leading edge of the plate up to transition. The transitional boundary layer determining the location of the onset of transition. Among it is known correlations [24] where the best one to the shear function is represented by Thwaites suggested a correlation for the local pressure gradient in both zero pressure gradient and in presence of pressure gradient in an unseparated boundary layer according to the following relation indicated by Sankar [23]

\[ \theta^2 = \frac{0.45v}{\nu^*} \int_0^x U_0^2 dx + C \]  

(7)

After \( \theta \) is found, the following constrains are used to compute the shape factor \( H \).

\[ H(\lambda) = \begin{cases} 
2.61 - 3.75\lambda + 5.24\lambda^2, & 0 \leq \lambda \leq 0.1 \\
2.472 + 0.0147\frac{1}{0.107+\lambda}, & -0.1 \leq \lambda \leq 0 
\end{cases} \]  

(8)

where,

\[ \lambda = \frac{\theta^2 dU_0}{v^* dx} \]  

(9)

Skin friction coefficient and displacement thickness can be calculated from the assumed one-parameter correlations [24]

\[ \tau_w = \frac{\mu U_0}{\theta} S(\lambda) \]  

(10)

\[ \delta^* = \theta H(\lambda) \]  

(11)

Thwaites suggested a correlation for \( S(\lambda) \) and \( H(\lambda) \), where the best one to the shear function is represented by

\[ S(\lambda) \approx (\lambda + 0.09)^{0.62} \]  

(12)

it is known

\[ \tau_w = \frac{1}{\rho} \frac{\partial U_0^2}{\partial \theta} C_f \]  

(13)

Then the friction coefficient can be calculated from the relation

\[ C_f = \frac{2\mu}{\rho U_0^2 \theta} (\lambda + 0.09)^{0.62} \]  

(14)

The transitional boundary layer
Numerous correlations have been existed in the literature determining the location of the onset of transition. Among those correlations, the one proposed by Abu-Ghannam and Shaw [3] which determines the starting of transition as a function of the free-stream turbulence level \( T_u \) and the local pressure gradient \( \lambda_p \). The empirical relation is based on extensive experimental data obtained in the highly turbulent flows likely to be encountered in turbomachinery, and was adopted for use in the present scheme. The function is given as

\[ Re_{BS} = 163 + \exp \left\{ f(\lambda_p) - \frac{f(\lambda_p)}{6.94} T_u \right\} \]  

(15)

where

\[ f(\lambda_p) = \begin{cases} 
6.91 + 12.75\lambda_p + 63.64(\lambda_p)^2, & \lambda_p < 0 \\
6.91 + 2.48\lambda_p - 12.27(\lambda_p)^2, & \lambda_p > 0 
\end{cases} \]  

(16)

For case of flat plate with zero pressure gradient the parameter \( \lambda \) goes to zero. To calculate the development of the boundary layer during transition in this method, it is required to determine first, the ending point of transition \( X_E \) and the properties of the boundary layer at this point.

The position of the end of transition \( X_E \) is specified by

\[ Re_{XE} = Re_{XS} + Re_L \]  

(17)

where,

\[ Re_L = 16.8(Re_{XS})^{0.8} \]  

(18)

and the momentum thickness Reynolds number \( Re_{OE} \) is determined from

\[ Re_{OE} = 540 + 183.5(Re_L \cdot 10^{-5} - 1.5)(1 - 1.4\lambda_p) \]  

(19)

The integral parameters at the end of transition \( (H_E, C_f) \) were related to momentum thickness Reynolds number \( (Re_{OE}) \) through the Ludwieg and Tillmann [25] correlation

\[ C_f = 0.246(10^{-0.67H_E}) Re_{OE}^{-0.26H_E} \]  

(20)

Furthermore, the following empirical relationship between \( H \) and \( Re_{OE} \) proposed by Goksel [26] is considered

\[ \log_e \frac{H}{H_{-1}} = 0.112(\log_e Re_{OE}) + 0.375 \]  

(21)

Specifying the momentum thickness Reynolds number at the end of transition, \( Re_{OE} \), equations (21) and (20) can be solved for the shape factor, \( H_E \), and friction coefficient, \( C_f \), at the end of transition respectively. Since \( Re_{BS} , H_S \) and \( C_f \) were known from Thwaites’s prediction, the following equations are used to calculate the development of the different integral parameters from the start to the end of transition region

\[ \theta' = \theta^{1.35} \]  

(22)

\[ H' = \sin \left( \frac{\pi}{2} \eta \right) \]  

(23)

\[ C_f' = 1 - \exp (-5.645 \eta^2) \]  

(24)

where \( \eta \) is a non-dimensional length parameter during transition defined as

\[ \eta = \frac{Re_{XE}-Re_{XS}}{Re_{XE}-Re_{XS}} \]  

(25)

The momentum thickness ratio during transition region \( \theta' \) is defined as

\[ \theta' = \frac{\theta - \theta_S}{\theta_E - \theta_S} \]  

(26)

\[ H' = \frac{H_S - H}{H_S - H_E} \]  

(27)

and \( C_f' \), friction coefficient during transition defined as
\[ C_f' = \frac{c_{f}\delta^{*}}{\delta} \]  

(28)

The turbulent boundary layer

In the turbulent region of the boundary layer, three popular integral methods were employed to predict the turbulent flow parameters. The first one was given by Head [27] and the second one proposed by Curle and Davies [28]. The aim of implementing the second method is to validate the entire code results for the three types of boundary layers (laminar, transitional and turbulent) with Abu-Ghannam and Shaw integral scheme. The last integral method employed for turbulent boundary layer has much greater capability and still simple to apply was developed by Moses [29]. In all methods, the boundary-layer parameters are required at an arbitrary starting point. Since the boundary layer at the end of transition (\(X_E\)) was found to be fully turbulent, thereby this point was considered as starting point of prediction.

In the Head method, Head suggested a new shape parameter \(H_t\) indicated by Sankar [23] and given by the following relation

\[ H_t = \frac{\delta - \delta^*}{\delta} \]  

(29)

Evolution of \(H_t\) along the boundary layer

\[ \frac{1}{U} \frac{d}{dx} (U \delta H_t) = 0.0306 (H_t - 3)^{-0.6169} \]  

(30)

Equation (30) and Von Karman Momentum Integral Equation (4) are solved by marching from transition location to trailing edge.

The empirical closure relations suggested by Head

\[ H_t = \begin{cases} 3.3 + 0.8234 (H - 1.1)^{-1.287}, & H \leq 1.6 \\ 3.3 + 1.5501 (H - 0.6778)^{-2.064}, & H > 1.6 \end{cases} \]  

(31)

Friction coefficient in Head method is calculated by Ludwig-Tillman relationship, Eq. (20)

Curle and Davies start from a power-law velocity profile and with analytical solution of empirical relationships they produced the following two equations for the calculation of the momentum thickness \(\theta\), and the shape factor \(H\), respectively [3],

\[ \theta (Re\theta)^{0.2} = \frac{0.106}{U_{\infty}^2} \int_0^x U_{\infty}^2 dx + C_1 \]  

(32)

and

\[ U_{\infty}^2 \left( \frac{1}{H - 1} - 4.762 \right) = 0.00307 \int_0^x \frac{U_{\infty}^2}{\theta \Re \theta^2} dx + C_2 \]  

(33)

\(C_1\) and \(C_2\) are constants of the integration given by

\[ C_1 = (\theta (Re\theta)^{0.2})_E \]  

(34)

\[ C_2 = \left( U_{\infty}^2 \left( \frac{1}{H - 1} - 4.762 \right) \right)_E \]  

(35)

where \(E\) refers to the end of transition.

The friction coefficient is calculated from the following function which tabulated against the shape factor \(H\) for a range of Reynolds number \(Re\theta\) between 10^6 and 10^8.

\[ G(H) = \frac{c_{f}(Re\theta)^{0.2}}{2} \]  

(36)

In Moses integral method, the development of turbulent boundary layer is calculated by dividing the layer into two or more strip [29].

The method uses a generalized form of the momentum integral equation, Eq. (37), written as

\[ \frac{u}{U_{\infty}} \frac{d}{dx} \left( Re_{\theta} \int_0^{\eta_1} \frac{u}{U_{\infty}} d\eta \right) - \frac{1}{U_{\infty}} \frac{d}{dx} \left( U_{\infty} Re_{\theta} \int_0^{\eta_1} \frac{u^2}{U_{\infty}^2} d\eta \right) = \frac{1}{\mu U_{\infty}} \frac{\eta_2 Re_{\theta} dU_{\infty}}{U_{\infty}} \frac{dU_{\infty}}{dx} \]  

(37)

where

\[ \eta = \frac{y}{\delta} \]  

(38)

For the required assumption of a velocity profile shape, Moses used an approximate form of the law of the wake verified by Schetz [30]

\[ \frac{u}{U_{\infty}} = 1 + \frac{\sqrt{\eta}}{\kappa} \ln(\eta) - \frac{2\eta}{\kappa} \left[ 1 - \frac{1}{2} W(\eta) \right] \]  

(39)

Turbulent shear has been modeled with an eddy viscosity model. Moses used the completely empirical eddy viscosity model

\[ \frac{\mu_T + \mu}{\mu_{\infty}} = 0.0225 + \frac{125}{Re_{\theta}} \]  

(40)

Two ordinary differential equations are derived by evaluating equation (37) at \(\eta_1 = 1\) and \(\eta_1 = 0.3\) together with the velocity profiles, Eq. (39) and eddy viscosity model, Eq. (40), for two unknowns, \(\delta(x)\) and \(C_T\).

More details of Moses integral method can be found in references [29] and [30].

4.2 Calculation of Thermal-Boundary Layers Parameters

Thermal laminar boundary layer

For the laminar boundary layer flows, Smith-Spalding integral method [31] has been implemented to evaluate the convective heat transfer along the flat plate surface.

The method is essentially a simplification and extension of the Eckert [32] method, and can be applicable both to planar or axisymmetric flows [31].

In this analysis, the heat transfer coefficient, \(h_{lam}\), is estimated by evaluating the laminar conduction thickness \(\Delta_{lam}^4\)

\[ \frac{U_{\infty}^2}{v} \Delta_{lam}^2 = 11.68 \int_0^x U_{\infty}^{1.87} dx + C \]  

(41)

and

\[ \Delta_{lam}^4 = \frac{k}{h_{lam}} \]  

(42)
Since the conduction thickness, $\Delta_{k, lam}$, is related to the heat flux $q_v$, through the local heat transfer coefficient

$$h = \frac{q_v}{(T_w - T_e)}$$  \hspace{1cm} (43)

Eq. (41) is converted to a dimensionless form, Nusselt number [24]

$$\frac{Nu}{\sqrt{Re}} = \frac{h L}{\sqrt{\nu L^2}} = \left[ \frac{y}{(y/\nu)} \right]^{-2.87} \left[ \int_0^y \left( \frac{y}{\nu} \right)^{2.87} dy \right]^{1/2}$$  \hspace{1cm} (44)

where $L$ is a reference length.

Other form of the previous relation suggested by Smith & Spalding [31] reads

$$\frac{Nu_X}{\sqrt{Re_X}} = \frac{1}{(\Delta_{k, lam}) \sqrt{Re_X}}$$  \hspace{1cm} (45)

and Stanton number

$$St = \frac{Nu}{Pr Re}$$  \hspace{1cm} (46)

The local $St_x$ in laminar boundary layer could be simplified to

$$St_{x, lam} = \frac{1}{(\Delta_{k, lam}) \sqrt{Re_X}}$$  \hspace{1cm} (47)

Another relation was developed by Ambrok [33] and was used by Silva et al. [17] to calculate the local convective heat transfer coefficient has been implemented in the program after rearrangement for $\Delta T = const.$

$$Nu_{lam} = 0.3Re_x \Delta T \left[ \int_{x_0}^{x_f} \frac{u \Delta T^2}{v} dx \right]^{1/2}$$  \hspace{1cm} (48)

The evaluation of “enthalpy flux” thickness, $\Delta_{2, lam}$ in laminar boundary layer is analyzed by using Ambrok [16] empirical relation

$$Re_{\Delta_2 lam} = \frac{0.03}{\Delta T} \left( \int_{x_0}^{x_f} \frac{u \Delta T^2}{v} dx \right)^{1/2}$$  \hspace{1cm} (49)

**Thermal transitional boundary layer**

To calculate the development of the thermal boundary layer during transition, the same technique of the momentum boundary layer analysis has been implemented. Since the location of the end of transition $X_g$ has already been determined, thereby this technique requires only specifying the thermal properties of the boundary layer at that point. As reported in transitional boundary layer, the boundary layer at the end of transition was fully turbulent so that the Stanton number ($St_e$) and the skin friction coefficient($C_f$), could be related through the Reynolds analogy relation [30]

$$St_e Pr^{2/3} = \frac{C_f}{2}$$  \hspace{1cm} (50)

The enthalpy thickness at the end of transition $\Delta_{2e}$ is calculated from Ambrok [33] and Kays and Crawford [34]

$$Re_{\Delta_2e} = \left( \frac{0.0125}{St_e} \right)^4 Pr^{2}$$  \hspace{1cm} (51)

The values of $St_e$ and $Re_{\Delta_2e}$ were used as initial condition for thermal turbulent boundary layer.

With $St_e$ and $Re_{\Delta_2e}$ known from Eq. (47) and Eq. (49) at specified location $X_g$ that determined by transition onset Eq. (15), equations (52) and (53) are used to calculate the development of thermal parameters from the start to the end of transition region.

$$\Delta_2' = \eta^{1.35}$$  \hspace{1cm} (52)

$$St' = 1 - \exp(-5.645\eta^2)$$  \hspace{1cm} (53)

where, the enthalpy flux thickness ratio during transition region $\Delta_2'$ is defined as

$$\Delta_2' = \frac{\Delta_2 - \Delta_{2s}}{\Delta_{2e} - \Delta_{2s}}$$  \hspace{1cm} (54)

and $St'$, heat transfer coefficient ratio during transition, defined by means of the Stanton number as

$$St' = \frac{St_e - St_S}{St_e - St_{SS}}$$  \hspace{1cm} (55)

The principles used in the program for the transition region is differ from that one used by Silva et al. [17]. Silva assumed $\Delta_{2, lr} = \Delta_{2, lam} = \Delta_{2, turb}$ and therefore he used the enthalpy thickness at the onset of transition $\Delta_{2, lr}$ calculated by Eq. (49) as initial condition for the thermal turbulent boundary layer $\Delta_{2, turb}$ Eq. (57), whereas in present scheme the initial condition for the thermal turbulent boundary layer was determined by Eq. (51) at the end of transition region.

This assumption by Silva et al. [17] satisfied the theory proposed by Dharwan and Narasimha [15] which indicated that, the turbulent boundary layer should be considered to begin at the starting of transition as reported by Mayle [2].

**Thermal turbulent boundary layer**

In fully turbulent boundary layer, the local convective heat transfer coefficient is evaluated according to [33] and [34] by Stanton number

$$St_{turb} = 0.0125Re_{\Delta_{2, turb}} Pr^{-1/2}$$  \hspace{1cm} (56)

The turbulent enthalpy thickness is evaluated by the following equation after rearrangement for constant temperature [17]

$$Re_{\Delta_{2, turb}} \Delta T = \left[ 0.0156 Pr^{-\frac{1}{2}} \mu^{-1} \int_{x_0}^{x_f} G \Delta T^{1.25} dx \right]^{0.8} + (Re_{\Delta_{2, turb}} \Delta T)^{1.25}$$  \hspace{1cm} (57)

**5 RESULTS AND DISCUSSION**

The first results figured in this work represent the validation of the predicted results of the present model with the results of the original model [3]. This validation has been done with both, model and experiment data of integral parameters, momentum thickness, $\theta$, friction coefficient, $C_f$, and shape factor, $H$. In case of using the integral method of Curle and Davies for turbulent
boundary layer within the present model, the model will be similar to that of Abu-Ghannam and Shaw model and therefore the assessment of the scheme performance can be verified. The results represented by Figures 2, 3 and 4 show fairly well agreements of these integral parameters which implies that, the code is well-structured and capable for predicting the boundary layers development on flat plate surface.

Since, the boundary layer flows subjected to freestream turbulence, thereby it is necessary to evaluate the effect of turbulence level on the onset of transition. Figure 5 compares the results of the present code with the code and experiments results of reference [3]. The results seem to be in good agreement with the predicted results of the reference.

The effect of freestream velocity upon the onset of transition is illustrated in Figure 6. The result is plotted for different turbulence intensities. The model represents satisfactory the influence of freestream velocity on the start of transition which implies an earlier transition onset while increasing freestream velocity at constant level of turbulence. This effect decreases gradually at high level of turbulence since there is only slight decay of the velocity curve. The Figure 6 also indicates that, the origination of a transition region on the flat plate for low level of turbulence required a high freestream velocity.

The results of this model also have been validated for momentum boundary layers by two-equation turbulence model, intermittency transport equations for modeling transition in boundary layers subjected to freestream turbulence and experimental data. The computational in-house models are, k-ε model of Launder and Sharma [1] and Menter et al. [7], where the experimental data is T3A. Both models and experimental results were published by Keerati Suluksa and Ekachai Juntasaro [35]. Specifications of the flow condition in two experimental works are summarized in Table 1.

The predicted results of the considered models are compared with the experimental data of the momentum thickness Reynolds number based, \( Re_{\theta} \), the skin friction coefficient, \( C_f \) and shape factor, \( H \). Figure 7 represent the development of the boundary layer in term of momentum thickness Reynolds number. In case of Blasius boundary layer flow, the boundary layer is laminar in the entrance region of the flat plate, and becomes transitional and then turbulent. The momentum thicknesses, from exact analysis, of laminar and turbulent boundary layers can be determined from \( \theta_{\text{lam}} = 0.664x/Re_\theta^{1/2} \) and \( \theta_{\text{turb}} = (7/72)0.166x/Re_\theta^{1/7} \) which are the upper and lower dash lines respectively. Given by [35], the transition in the experimental data T3A case starts at Reynolds number of 1.35 \( \cdot \) 10^5 corresponding to the momentum thickness Reynolds number of 272 and ends at the Reynolds number of 3.09 \( \cdot \) 10^5 or the momentum thickness Reynolds number of 628.

Skin friction coefficient is indicted by Figure 8, which plays a vital role in indicating the starting and ending points of transition. In addition, the determination of the growth rate characteristic of transition leading to the length of transition to be found. The analytic skin friction coefficient for laminar and turbulent flows in case of the flat plate boundary layer flow with zero pressure gradient condition can be determined from \( C_f = 0.664/Re^{1/2} \) and \( C_f = 0.027/Re^{1/7} \) respectively and displayed by the dash lines in Figure 8. The variation of the friction coefficient within the transition region indicates the growth rate and length of transition.

Figure 9 displays the shape factor which describes the influence of the freestream turbulence length scale on the transition. Moreover, it indicates if the boundary layer is separated or has the tendency to separate. However, in Blasius boundary layer case considered here, the shape factor indicates the region where the boundary layer tends to be turbulent.

In case of k-ε model of Launder and Sharma [1], since this model is known to be the best performer of low Reynolds number turbulence models in predicting transition [7] therefore the momentum thickness Reynolds number is expected to be located in the transition region. Due to this result, the deviation of its profile from the laminar line does not appear at the leading edge (Fig. 7) which verifies the ability of the model in predicting the transitional flow. The model computes the laminar boundary layer from the leading edge till the transition onset and then continues in computing the transitional boundary layer with a rapid growth rate of transition to turbulence. This results in a shorter transition length when compared with the experimental data.

With the model of Menter et al. [7], this transition model predicts fairly well the development of the boundary layer (Fig. 7). In comparison of the computed skin friction coefficient with the experimental data, the model is able to handle satisfactory transition onset and transition length. In comparison of the shape factor, the results agree well with the experimental data. The model has the capability to simulate the diffusion of turbulent eddies into the underlying laminar boundary layer as mentioned by [35]. The reason is that the intermittency factor at the upstream location of transition onset is set to unity which reveals in well predicted decrease of the measured shape factor starting from the leading edge.

With the present model, the deviation of the momentum thickness Reynolds number profile from the laminar line shows an early onset of transition. In the model of Abu-Darag, the transition starts at Reynolds numbers of 0.93 \( \cdot \) 10^5, corresponding to the momentum thickness Reynolds numbers of 198 and end at \( Re_{\theta} = 2.46 \cdot 10^5 \) corresponds to 549.4. The variation of the skin friction coefficient during transition region indicates this earlier origination of transition which approximately close to the result of k-ε model of Launder and Sharma [1] but with slow growth rate of the transition. Since the later model was considered to be suitable for use in simulating the transition at high level of turbulence [35], thereby this earlier-prediction between the present integral model
results and the results of T3A and Menter et al. [7] it may
be returns to either the different in length scale of
turbulence, $L_x$, in case of the experimental works or due
to the different in empirical correlations used by both
models, Abu-Darag and Menter et al. [7] in case of the
computational exercise. These empirical correlations
are presented in Table 2 and used to predict the onset and
length of transition region in both models. Since the flow
with zero pressure gradient, the model predicts a constant
shape factor at the laminar value of 2.6 from the leading
dge toward the transition onset. Although the model
shows an early decay of the shape factor during the
transition region which is very close to the model of
Lauder and Sharma [1], it is successfully able to detect
the fully turbulent behavior.

To evaluate the performance of the present work for
thermal boundary layer development, two parameters are
illustrated, convective heat transfer coefficient and
enthalpy flux thickness.

Figure 10 represents the predicted non-dimensional heat
transfer coefficient, Stanton number, $St$, against the
experimental works of M. F. Blair [36] and Ki-Hyeon
and Eli Reshotko [37]. Details of both experimental flow
conditions are tabulated in Table 1. Analytical solutions
of both laminar and fully turbulent boundary layer flows
over flat plate with constant surface temperature
suggested by Kays and Crawford [34] are $St_{lam} =
0.332Pr^{-2/3}Re_x^{1/2}$ and $St_{turb} = 0.0295Pr^{-0.4}Re_x^{-0.2}$
respectively and included in the figure. The results of
both experimental data are in very good agreement as
shown in the figure and as reported by [37]. In
comparison of the integral calculated heat transfer
coefficient, $St$, with the experimental data, the code gives
a little early onset of transition, but with a satisfactory
length of transition. This small difference in transition
origination cannot implies that the transition with Abu
Darag model is occurs upstream than other references as
with the case of T3A. The reason is that, this difference is
approximately around 3.5 cm which exactly represents
the unheated starting length, $x_0$, in both experimental
works. The development of the thermal boundary layer in
case of present model starts at the leading edge as
represented in Figure 11, while for both references it
starts after small unheated distance. Exact solution for the
thermal laminar boundary layer in Figure 11 is plotted
according to the relation given by Kays and Crawford
[34] as $St_{lam}Pr^{4/3} = 0.2205/Re_{\theta_\lambda}$ while for thermal
turbulent boundary layer Eq. (55) is used. The correction
of this unheated starting length which was used in the two
references is $St_{lam} = 0.453Pr^{-2/3}Re_x^{-1/2}[1
- (x_0/x)^{3/4}]-1/3$, given by Kays and Crawford [34] and
plotted in the same figure. Moreover Blair [36] was
verified that the transition onset in his work is in a good
agreement with some conducted experiments. Among
them, the experimental investigation of Dryden [38], Hall
and Hislop [39] and Schubauer and Skramstad [40].
These experimental works themselves were used by Abu-
Ghannam and Shaw to validate their results for the start of
transition region. In other words the results of M. F. Blair
[36] and, Ki-Hyeon and Eli Reshotko [37] should be
fairly agree with the result of the present model after
correction for unheated starting length.

6 CONCLUSIONS

The proposed integral methods used within the present
program are approved good performance and well-
structured scheme for both momentum and thermal
boundary layers with zero pressure gradient over flat plate
surface subjected to low level of turbulent intensity and
high Reynolds number. Validation of the code with the
code written by Abu-Ghannam and Shaw [3] is performed
and the results of integral parameters within the
transitional boundary layer ensured fairly good
agreement. The effect of turbulence level and Reynolds
number on the onset of transition is checked and their
results are analyzed.

In case of thermal boundary layer subjected to free-stream
 turbulence intensity, the result of the heat transfer
coefficient, $St$, is validated with the recommended
experimental work of M. F. Blair [36] and Ki-Hyeon
Sohn and Eli Reshotko [37]. Analysis for level of
turbulence intensity of 2.4% is performed and the results
showed well predicted Stanton number for case of the
moderate level of turbulence. The development of
isothermal boundary layer is plotted in terms of Reynolds
number based on enthalpy flux thickness $Re_{\theta_\lambda}$.

Validation of the program with numerical software results
is also performed. In computation of Abu-Darag model
with the transition model with intermittency transport
equations of Menter et al. [7], the model gives a too early
onset of transition and an unsatisfactory transition length
for case of high free-stream turbulence intensity and low
Reynolds number, but gives a good predicted results for
the boundary layer development when compared with the
$k-\varepsilon$ turbulence model of Launder and Sharma [1].

<table>
<thead>
<tr>
<th>Case</th>
<th>$U_\infty$ (m/s)</th>
<th>$Tu$ (%)</th>
<th>$Re_{\theta_\lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abu-Ghannam and Shaw</td>
<td>22</td>
<td>1.25</td>
<td>5.63 · $10^4$</td>
</tr>
<tr>
<td>T3A</td>
<td>5.4</td>
<td>3.35</td>
<td>6.12 · $10^4$</td>
</tr>
<tr>
<td>M. F. Blair</td>
<td>30.3</td>
<td>2.5</td>
<td>-</td>
</tr>
<tr>
<td>Ki-Hyeon and Eli Reshotko</td>
<td>30.5</td>
<td>2.4</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Flow characteristics

<table>
<thead>
<tr>
<th>Model</th>
<th>Empirical Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abu-Ghannam and Shaw</td>
<td>$Re_{\theta_\lambda} = 163 + \exp (6.91 - Tu)$</td>
</tr>
<tr>
<td>Menter et al. [7]</td>
<td>$Re_{\theta_\lambda} = 80.73(Tu + 0.6067)^{1.027}$</td>
</tr>
</tbody>
</table>

Table 2: Empirical correlation used in models
Figure 2: Predicted momentum thickness Reynolds number in a zero pressure gradient transitional boundary layer

Figure 3: Predicted skin friction coefficient in a zero pressure gradient transitional boundary layer

Figure 4: Predicted shape factor in a zero pressure gradient transitional boundary layer

Figure 5: Momentum thickness Reynolds number at start and end of transition for zero pressure gradient, \( \frac{dp}{dx} = 0 \)

Figure 6: Influence of freestream velocity on transition onset for different turbulence levels

Figure 7: Predicted momentum thickness Reynolds number in transitional boundary layer subjected to moderate turbulence intensity
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