TOPICS IN MATHEMATICAL MODELLING OF LIFE SCIENCES PROBLEMS

PROBLEME ACTUALE ÎN MODELAREA MATEMATICĂA PROBLEMELOR DIN ŞTIINŢELE VIEŢII

Topics
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of
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Topics in Mathematical Modelling of Life Sciences Problems

Mathematical modelling of overland flow and soil erosion in the presence of vegetation

by Stelian Ion, Anca Veronica Ion, Dorin Marinescu, Stefan Gicu-Cruceanu [1]

Abstract: We consider the phenomena of soil erosion and transport due to the rain, in the presence of vegetation and aim to construct a partial differential equations based model. We systematically obtain mass and momentum balance equations for the water flow in the presence of sources (water input from rain and/or water loss by infiltration). Such equations were present in the literature, but were not explicitly obtained, their form being rather extrapolated from the classical Saint Venant equations. Then we present equations for the sediment transport without and with vegetation.

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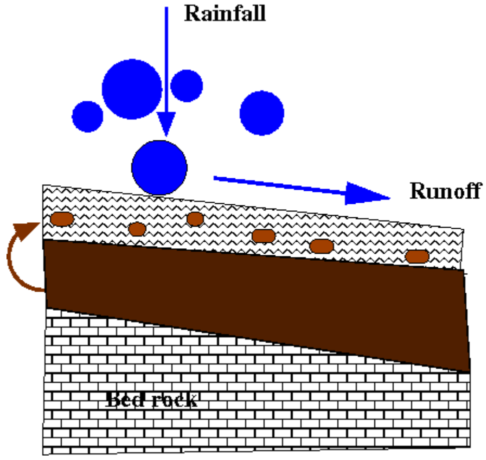


Fig. 1. Erosion and transport soil particles along a hill slope.

1. Physical processes

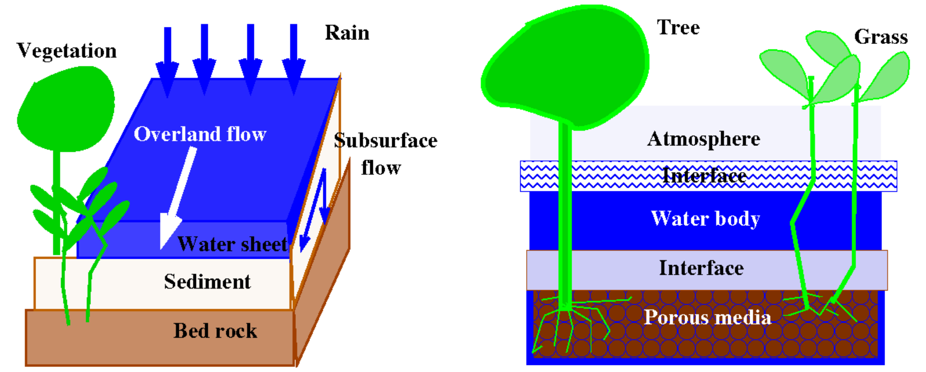
Roughly speaking, soil erosion means the moving of a certain quantity of soil from a point to another point. The soil can be carried by the wind, water, or it can move due to gravitational force on steep land. The soil erosion produced by water movement can be divided in splash erosion, interill erosion, rill erosion and gully erosion.

All these processes imply many factors that are very hard to be quantified or measured, the readers interested in the subject are refered to, [3], [11],[12], [13], [27].

In this paper we restrict to the soil erosion produced by the water sheet flow. The process of water flow on a hill slope, overland flow, is a complex phenomenon (a sketch of it being drawnin the fig. 2) involving various media, as well as interactions of different sub-processes, [17], [16], [28].

The media implied are atmosphere, water sheet, soil and vegetation. The water comes from rain, infiltrates in the soil and accumulates on the soil surface. The plants retain some quantity of water direct from the rain, absorb some quantity from the soil, modify the hydraulic soil properties and induce a resistance in the water flowing. To model the whole process one needs to make some simplifications and select certain dominant phenomena that are supposed to be important. The physics framework of the mathematical that we will present consists in:

Atmosphere. Acts as an external medium. The only phenomenon that we



Overland flow — Media implied in overland flow.

Fig. 2. The complex process of the water flow on hill slope covered by plants.

take into account is the rain, the rain drops accumulate on the sheet of water and produce the increase of the water depth.

Water sheet. It is a continuum medium whose state is described by the water depth and the velocity field.

Soil. It is considered as a porous medium. The surface water infiltrates in the soil and the water detaches and move particle from soil. The quantities that count in the surface flow is water content of the porous media.

Vegetation. The vegetation is taken into account by: an interception function in the water balance in the water sheet, a sink function in water balance in porous media, and modification of the constitutive functions of the soil hydraulics. As concerning the influence of the plant on the dynamics of the water flow in the water sheet we postpone any comment until the section devoted to sheet water.

Another concept that proves to be useful in modeling is the interface of two different media. Generally speaking an interface is an idealization of a narrow space domain where some phases transition or processes transition takes place.

In our mathematical model of overland flow we consider the following two interfaces:

Atmosphere - water sheet interface. It separates the atmosphere from the water bulk. On this surface rain drops accumulate and they exert a drag force on the water bulk. Mathematically one must impose jump conditions on the mass density and momentum density.

Water sheet - porous media interface. It separates the water bulk from porous media. On this surface water infiltrates from water bulk to soil and sediment is detached from and deposited on soil. Here, one needs jump conditions that relate the variables in the water bulk and porous media.

2. Sheet of water flow

The accumulation of water on the surface of the soil is a process that involves rain and infiltration into soil, the rain drops produce a layer of water if rain rate is greater than infiltration rate. This stratum of exceeding water moves on the soil surface down the hill. This flow was modelled in [26] and the flow described by this model is usually named hortonian flow. The water depth and velocity distribution are two physical quantities that are determinant in the process of the soil erosion. There exist several models of water flow on hill slope and many of them are process oriented [5], [8], [5], [14], [15]. As a consequence, it is very hard to extrapolate them to another context or to establish what is generally common for all flow processes on hill slope.

We find that the Saint Venant equations can be considered as common ground for most models used for hill slope flow. The Saint Venant equations in turn are obtained from Navier Stokes equations by using an asymptotic analysis and a mediation process, [20], [21], [22], [23], [24]. Here, we present a variant of these equations for overland flow.

The Saint Venant system of equations governs the motion of a fluid that exhibits a small variation in a preferential direction. One assumes that there exists a plane surface, the support plane, such that the hill surface slightly deviates from that support plane. The coordinate axis Ox^3, Ox^1 and Ox^2 are the normal direction and tangent direction to the support plane, respectively.

The flow domain is given by the accumulated water on the hill surface, and can be mathematically written as

$$\Omega = \{\eta(x^1, x^2, t) < x^3 < \xi(x^1, x^2, t) \,\big|(x^1, x^2) \in B \in \mathbb{R}^2\}$$

The surface $x^3 = \xi(x^1, x^2, t)$ separates the water flow domain from atmosphere and $x^3 = \eta(x^1, x^2, t)$ represents the hill surface. The time variation of the interface $x^3 = \xi(x^1, x^2, t)$ is due to the motion of the ponded water on the surface and to the water gain from the rain. This second cause precludes one to treat the interface as a material surface. The outward normal to the

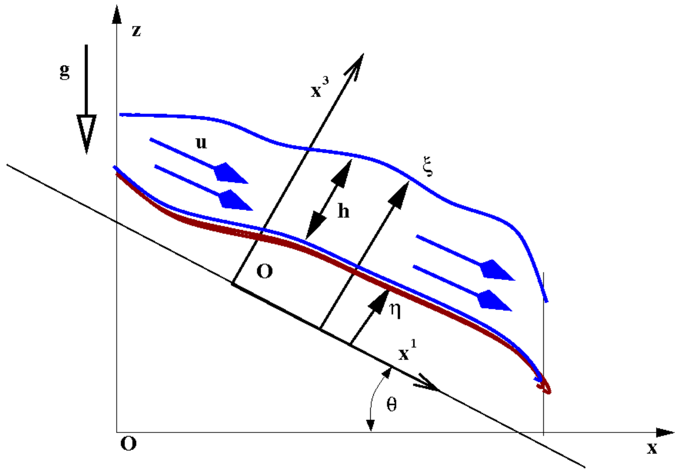


Fig. 3. The physical configuration of the water flow on a hill.

upper surface is given by

$$\boldsymbol{N} = \begin{pmatrix} -\partial_1 \xi \\ -\partial_2 \xi \\ 1 \end{pmatrix}, \ \boldsymbol{n} = \frac{1}{\sqrt{1 + |\nabla \xi|^2}} \boldsymbol{N},$$

and the normal velocity of the surface is given by

$$u_N = \partial_t \xi, u_n = \frac{1}{\sqrt{1 + |\nabla \xi|^2}} u_N,$$

Similar formulas hold for the wall surafce

$$\begin{pmatrix} \partial_1 \eta \\ \partial_2 \eta \\ -1 \end{pmatrix}, \ \boldsymbol{n} = \frac{1}{\sqrt{1 + |\nabla \eta|^2}} \boldsymbol{N},$$

and the normal velocity of the surface is given by

$$u_N = -\partial_t \eta, u_n = \frac{1}{\sqrt{1 + |\nabla \eta|^2}} u_N.$$

Let $V(t) \in \Omega$ be an arbitrary domain. The integral form of the balance equations for any quantity with the density ψ read as:

$$\int\limits_{V(t_2)} \psi \mathrm{d}x - \int\limits_{V(t_1)} \psi \mathrm{d}x + \int\limits_{t_1}^{t_2} \int\limits_{\partial V(t)} \Phi^i_\psi n_i \mathrm{d}\sigma \mathrm{d}t + \int\limits_{t_1}^{t_2} \int\limits_{\partial V(t)} \psi \left(v_n - u_n\right) \mathrm{d}\sigma \mathrm{d}t = \\ = \int\limits_{t_1}^{t_2} \int\limits_{V(t)} f_\psi \mathrm{d}x \mathrm{d}t, \tag{2.1}$$

where f_ψ is the production density and Φ_ψ represents the flux density associated to the mechanical quantity with density ψ. The normal $\boldsymbol{N}$ to the surface is outward orientated, $v_N = v^i N_i, v_n = v^i n_i$ denotes the normal components of the fluid velocity to the surface ∂V and u_n or u_n is the normal component of the surface velocity.

The global form of the balance equation allows one to obtain the jump conditions on the interface between two different continua. If one chooses the domain $V(t)$ such that it intersects the interface and then it shrinks continuum from both size of the interface to it, one obtains

$$\int\limits_\Sigma \left[\left|\Phi^i_\psi n_i\right|\right] \mathrm{d}\sigma + \int\limits_\Sigma \left[\left|\psi \left(v_n - u_n\right)\right|\right] \mathrm{d}\sigma = 0, \tag{2.2}$$

where Σ is an arbitrary portion of the interface and the $[|\phi|]$ stands for the jump of the quantity ϕ.

From the integral form of the jump equation (2.2) one obtains the local form

$$\left[\left|\Phi^i_\psi n_i\right|\right] + \left[\left|\psi \left(v_n - u_n\right)\right|\right] = 0. \tag{2.3}$$

Chose $V = \{h_1(x^1, x^2, t) < x^3 < h_2(x^1, x^2, t) \left|(x^1, x^2) \in D\right.\}$ where D is the rectangle $\{(x^1, x^2) \left|x^1_b < x^1 < x^1_u, x^2_b < x^2 < x^2_u\right.\}$ and $h_1(x, t)$ and $h_2(x, t)$ are two arbitrary surfaces.

The integrals on the boundary $\partial V(t)$ can be written as

$$\int\limits_{\partial V(t)} \psi \left(v_n - u_n\right) \mathrm{d}\sigma = \int\limits_D \left(\partial_a \int\limits_{h_1(x^1,x^2,t)}^{h_2(x^1,x^2,t)} \psi v^a(x^1, x^2, x^3, t) \mathrm{d}x^3\right) \mathrm{d}x^1 \mathrm{d}x^2 \\ + \int\limits_D \psi \left(v_N - u_N\right)\Big|_{x^3 = h_2} \mathrm{d}x^1 \mathrm{d}x^2 + \int\limits_D \psi \left(v_N - u_N\right)\Big|_{x^3 = h_1} \mathrm{d}x^1 \mathrm{d}x^2, \tag{2.4}$$

$$
\begin{aligned}
\int\limits_{\partial V(t)} \Phi^i_\psi n_i \mathrm{d}\sigma = \int\limits_D \left(\partial_a \int\limits_{h_1(x^1,x^2,t)}^{h_2(x^1,x^2,t)} \Phi^a_\psi \mathrm{d}x^3 \right) \mathrm{d}x^1\mathrm{d}x^2 \\
+ \int\limits_D \Phi^i_\psi N_i\big|_{x^3=h_2} \mathrm{d}x^1\mathrm{d}x^2 + \int\limits_D \Phi^i_\psi N_i\big|_{x^3=h_1} \mathrm{d}x^1\mathrm{d}x^2.
\end{aligned}
\tag{2.5}
$$

The balance equation, for this choice of the domain, can be written as

$$
\begin{aligned}
&\int\limits_D \int\limits_{h_1(x^1,x^2,t_2)}^{h_2(x^1,x^2,t_2)} \psi \mathrm{d}x - \int\limits_D \int\limits_{h_1(x^1,x^2,t_1)}^{h_2(x^1,x^2,t_1)} \psi \mathrm{d}x + \\
&+ \int\limits_{t_1}^{t_2} \int\limits_D \left(\partial_a \int\limits_{h_1(x^1,x^2,t)}^{h_2(x^1,x^2,t)} \psi v^a(x^1,x^2,x^3,t) \mathrm{d}x^3 \right) \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}t + \\
&+ \int\limits_{t_1}^{t_2} \int\limits_D \left(\partial_a \int\limits_{h_1(x^1,x^2,t)}^{h_2(x^1,x^2,t)} \Phi^a_\psi \mathrm{d}x^3 \right) \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}t + \\
&+ \int\limits_{t_1}^{t_2} \int\limits_D \psi \left(v_N - u_N\right)\big|_{x^3=h_2} \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}t + \int\limits_{t_1}^{t_2} \int\limits_D \psi \left(v_N - u_N\right)\big|_{x^3=h_1} \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}t + \\
&+ \int\limits_{t_1}^{t_2} \int\limits_D \Phi^i_\psi N_i\big|_{x^3=h_2} \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}t + \int\limits_{t_1}^{t_2} \int\limits_D \Phi^i_\psi N_i\big|_{x^3=h_1} \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}t = \\
&= \int\limits_{t_1}^{t_2} \int\limits_D \int\limits_{h_1(x^1,x^2,t)}^{h_2(x^1,x^2,t)} f_\psi \mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{2.6}
$$

(2.6) is the basic equation for deducing the mediate form of the balance equation. Since t_2 and t_1 are arbitrary, then

$$\int_D \partial_t \int_{h_1}^{h_2} \psi(x,y,t)\mathrm{d}y\mathrm{d}x + \int_D \left(\partial_a \int_{h_1}^{h_2} \psi v^a(x,y,t)\mathrm{d}y + \partial_a \int_{h_1}^{h_2} \Phi_\psi^a(x,y)\mathrm{d}y \right) \mathrm{d}x +$$
$$+ \int_D \psi \left(v_N - u_N\right)\big|_{x^3=h} \mathrm{d}x + \int_D \psi \left(v_N - u_N\right)\big|_{x^3=g} \mathrm{d}x +$$
$$\int_D \Phi_\psi^i N_i\big|_{x^3=h} \mathrm{d}x + \int_D \Phi_\psi^i N_i\big|_{x^3=g} \mathrm{d}x = \int_D \int_{h_1}^{h_2} f_\psi(x,y)\mathrm{d}y\mathrm{d}x, \tag{2.7}$$

and

$$\partial_t \int_{h_1(x,t)}^{h_2(x,t)} \psi(x,y,t)\mathrm{d}y + \partial_a \int_{h_1(x,t)}^{h_2(x,t)} \psi v^a(x,y,t)\mathrm{d}y + \partial_a \int_{h_1(x,t)}^{h_2(x,t)} \Phi_\psi^a(x,y)\mathrm{d}y +$$
$$+ \psi\left(v_N - u_N\right)\big|_{x^3=h_2} + \psi\left(v_N - u_N\right)\big|_{x^3=h_1} + \Phi_\psi^i N_i\Big|_{x^3=h_2} + \Phi_\psi^i N_i\Big|_{x^3=h_1} =$$
$$\int_{h_1(x,t)}^{h_2(x,t)} f_\psi(x,y)\mathrm{d}y. \tag{2.8}$$

2.1. Mass balance equation

The mass balance equation results from the general equation (2.6) by considering $\psi = \rho$. We assume that the mass density is a constant function, $f_\psi = 0$ and $\Phi_\psi = 0$. We get

$$\partial_t \int_{h_1(x,t)}^{h_2(x,t)} \mathrm{d}y + \partial_a \int_{h_1(x,t)}^{h_2(x,t)} v^a(x,y,t)\mathrm{d}y = -\left(v_N - u_N\right)\big|_{x^3=h_2} - \left(v_N - u_N\right)\big|_{x^3=h_1}. \tag{2.9}$$

By considering $h_2(x,t) = \xi(x^1,x^2,t)$ and $h_1(x,t) = \eta(x^1,x^2,t)$ one gets

$$\partial_t h + \partial_a \int_\eta^\xi v^a \mathrm{d}y = r - i, \tag{2.10}$$

where $h = \xi(x^1,x^2,t) - \eta(x^1,x^2,t)$ is water depth, r is rain rate and i infiltration rate.

2.2. Momentum balance equations

Let us consider the water to be a viscous fluid. The stress tensor is given by

$$\frac{\sigma_{ij}}{\rho} = -p\delta_{ij} + \tau, \tag{2.11}$$

where p is the pressure field and τ is the viscosity part of the stress tensor given by

$$\tau_{i,j} = \nu(\partial_i v_j + \partial_j v_i).$$

The momentum balance equations are obtained by considering $\psi = \rho \boldsymbol{v}$. The density flux Φ is the stress tensor $-\boldsymbol{\sigma}$ and the production density f is the density of gravity force.

$$\begin{aligned} \partial_t \int\limits_{h_1(x,t)}^{h_2(x,t)} v^i \mathrm{d}y + \partial_a \int\limits_{h_1(x,t)}^{h_2(x,t)} v^i v^a \mathrm{d}y + \partial_a \int\limits_{h_1(x,t)}^{h_2(x,t)} \left(p\delta^{ia} - \tau^{ia}\right) \mathrm{d}y + \\ + \left(v^i\left(v_N - u_N\right) + pN^i - \tau^{ij}N_j\right)\big|_{x^3=h_2} + \\ + \left(v^i\left(v_N - u_N\right) + pN^i - \tau^{ij}N_j\right)\big|_{x^3=h_1} = -ge_i(h_2 - h_1), \end{aligned} \tag{2.12}$$

e_i represent the components of the versor of gravitational force in the new coordinate system $Ox^1x^2x^3$. The jump relations on the interface $x^3 = \xi$ that separates the water and atmosphere have the form

$$\rho(v^i(v_N - u_N) + pN^i - \tau^{ij}N_j)\big|_{x^3=\xi}^{\text{water}} = \rho(v^i(v_N - u_N) + pN^i - \tau^{ij}N_j)\big|_{x^3=\xi}^{\text{air}}. \tag{2.13}$$

To model the atmosphere-water body interaction one assumes that the state of atmosphere affects the state of the layer of water but not inverse. Consequently, all terms in r.h.s of the compatibilty relation are known. The term

$$\rho(v^i(v_N - u_N))\big|_{x^3=\xi}^{\text{air}}$$

quantifies the tranfer of the momentum from the atmosfere to the water body. Even though the term is different from zero if there is no rain, it can be negleted for moderate rain intensity and in the present work we make this assumption.

The terms

$$\rho\tau^{ij}N_j\big|_{x^3=\xi}^{\text{air}}$$

represent the traction that is exerted by the atmosphere on the body. The term is important in the case of storm, but for moderate wind velocity one can neglect it.

Henceforth, we consider

$$(v^i(v_N - u_N) + pN^i - \tau^{ij}N_j)\big|_{x^3=\xi}^{\text{water}} = p_a N^i \tag{2.14}$$

where $p_a = p_{air}/\rho_{water}$

2.3. Rescaled equations. Hydrostatic approximation of the pressure field

The hydrostatic approximation refers to the linear dependence of the pressure field with respect to the x^3 coordinate. As in [20] we rescale the Ox^3 component of the momentum equation by considering the following characteristic dimensions:
(a) a characteristic length L in the direction of the Ox^1 and Ox^2 axes and another characteristics length H in the direction Ox^3. One assumes that the characteristic length L is larger than the characteristic length H. As consequence the parameter

$$\epsilon = \frac{H}{L}$$

is a small quantity.
(b) the characteristic speeds V and W in the parallel direction and normal direction respectively. Another assumption regarding the magnitude of the kinetics of the flow is

$$W = \epsilon V.$$

Using the characteristic dimensions one rescales the coordinates, time velocities and pressure as follows:

the space and time

$$x^a = L\widetilde{x}^a, a = 1, 2; x^3 = H\widetilde{x}^3, t = \frac{L}{V}\widetilde{t},$$

the velocity and pressure field

$$v^a = V\widetilde{v}^a, v^3 = W\widetilde{v}^3, p = V^2\widetilde{p},$$

the upper surface (rescaled by using H)

$$\xi = H\widetilde{\xi},$$

the viscous part of the stress tensor can be written

$$\tau = V^2\widetilde{\tau},$$

the components of the rescaled stress are given by

$$\widetilde{\tau}_{ab} = \frac{1}{\mathrm{Re}}(\widetilde{\partial}_a \widetilde{v}_b + \widetilde{\partial}_b \widetilde{v}_a), \widetilde{\tau}_{a3} = \frac{1}{\mathrm{Re}}\left(\epsilon \widetilde{\partial}_a \widetilde{v}_3 + \frac{1}{\epsilon}\widetilde{\partial}_3 \widetilde{v}_a\right), \widetilde{\tau}_{33} = \frac{2}{\mathrm{Re}}\widetilde{\partial}_3 \widetilde{v}_3, \quad (2.15)$$

where

$$\frac{1}{\mathrm{Re}} := \frac{\nu}{VL}$$

is the Reynolds' number.

To obtain the hydrostatic approximation of the pressure field we use equation (2.12). We set $h_2(x^1, x^2, t) = \xi(x^1, x^2, t)$ and for a given z between $\eta(x^1, x^2, t)$ and ξ we set $h_1(x^1, x^2, t) = z$. We use the relation (2.14), then rescale all terms and obtain

$$\begin{aligned} \epsilon^2 \left(\partial_{\widetilde{t}} \int_{\widetilde{z}}^{\widetilde{\xi}} \widetilde{v}^3 \mathrm{d}y + \widetilde{\partial}_a \int_{\widetilde{z}}^{\widetilde{\xi}} \widetilde{v}^3 \widetilde{v}^a \mathrm{d}y \right) - \epsilon \widetilde{\partial}_a \int_{\widetilde{z}}^{\widetilde{\xi}} \widetilde{\tau}^{a3}(\epsilon) \mathrm{d}y - \\ - \epsilon^2 \left(\widetilde{v}^3\right)^2 + \widetilde{\tau}^{33}(\widetilde{x}^a, \widetilde{z}) + \widetilde{p}_a - \widetilde{p}(\widetilde{x}^a, \widetilde{z}) = -g\frac{H}{V^2} e_3 \left(\xi - \widetilde{z}\right) \end{aligned} \quad (2.16)$$

Taking the limit with $\epsilon \to 0$, one obtains

$$\begin{aligned} -\frac{1}{\mathrm{Re}}\widetilde{\partial}_a \widetilde{v}^a(\widetilde{x}^1, \widetilde{x}^2, \widetilde{\xi}) + \frac{1}{\mathrm{Re}}\widetilde{\partial}_a \widetilde{v}^a(\widetilde{x}^1, \widetilde{x}^2, \widetilde{z}, t) + \frac{2}{\mathrm{Re}}\widetilde{\partial}_3 \widetilde{v}^3(\widetilde{x}^a, \widetilde{z}, t) - \\ + \widetilde{p}_a - \widetilde{p}(\widetilde{x}^1, \widetilde{x}^1, \widetilde{z}, t) = -\frac{1}{\mathrm{Fr}^2} e_3 \left(\xi - \widetilde{z}\right). \end{aligned} \quad (2.17)$$

where $\mathrm{Fr} = V/\sqrt{gH}$ is the Froude number. If one uses the incompressibility condition of the water, $\widetilde{\partial}_i \widetilde{v}^i = 0$, the last equation can be rewritten as

$$\begin{aligned} \widetilde{p}(\widetilde{x}^1, \widetilde{x}^2, \widetilde{z}, t) = \widetilde{p}_a + \frac{1}{\mathrm{Fr}^2} e_3 \left(\xi - \widetilde{z}\right) - \\ -\frac{1}{\mathrm{Re}}\left(\widetilde{\partial}_a \widetilde{v}^a(\widetilde{x}^1, \widetilde{x}^2, \widetilde{\xi}(\widetilde{x}^1, \widetilde{x}^2, t), t) + \widetilde{\partial}_a \widetilde{v}^a(\widetilde{x}^1, \widetilde{x}^2, \widetilde{z}, t)\right). \end{aligned} \quad (2.18)$$

By rewriting the scaled equation of the pressure in the original variable, one has

$$p(\boldsymbol{x}, t) = p_a + g e_3(\xi(x^1, x^2, t) - x^3) + 1/2(\tau^a_{s,\,a}(x^1, x^2, t) + \tau^a_a(\boldsymbol{x}, t)) \quad (2.19)$$

where

$$\tau^a_{s,\,a} = 2\nu \partial_a v^a(x^1, x^2, \xi(x^1, x^2, t), t), \; \tau^a_a(\boldsymbol{x}, t) = 2\nu \partial_a v^a(\boldsymbol{x}, t).$$

By neglecting the viscous stress, the pressure field is given by

$$p(\boldsymbol{x}, t) = p_a + g e_3(\xi - x^3). \quad (2.20)$$

Hydrostatic approximation of the plane parallel momentum equation. For parallel components of the momentum balance equation we use again the equation (2.12). Set the $h_2(x^1, x^2, t) = \xi(x^1, x^2, t)$, $h_1(x^1, x^2, t) = \eta(x^1, x^2, t)$ and let

$$h = \xi(x^1, x^2, t) - \eta(x^1, x^2, t),$$

be the water depth. We get

$$\begin{aligned}\partial_t \int_\eta^\xi v^b \mathrm{d}y + \partial_a \int_\eta^\xi v^b v^a \mathrm{d}y - \partial_a \int_\eta^\xi \tau^{ab} \mathrm{d}y + \partial_b \int_\eta^\xi p \mathrm{d}y \\ + p_a N^b\big|_{x^3=\xi} + pN^b\big|_{x^3=\eta} + v^b\left(v_N - u_N\right)\big|_{x^3=\eta} - \tau^{bi} N_i\big|_{x^3=\eta} \\ = -ge_b\left(\xi(x^1, x^2, t) - \eta(x^1, x^2, t)\right).\end{aligned} \tag{2.21}$$

Using the hydrostatic approximation of the pressure, one has

$$\partial_a \int_\eta^\xi p\mathrm{d}y + pN^b\Big|_{x^3=\xi} + pN^b\Big|_{x^3=\eta} = h\partial_b p_a + ge_3\partial_b\left(\frac{h^2}{2}\right) + ge_3 h\partial_b\eta + \Sigma_b$$

where

$$\Sigma_b = 1/2\partial_b\left(h\tau^a_{s,\,a}\right) + 1/2\left(\tau^a_{s,\,a} - \tau^a_a|_{x^3=\eta}\right)\partial_b\eta - 1/2\partial_b \int_\eta^\xi \tau^a_a(x^a, y)\mathrm{d}y$$

is the contribution of the viscosity to the pressure field.

Again by neglecting the viscous terms, one has

$$\partial_a \int_\eta^\xi p\mathrm{d}y + pN^b\Big|_{x^3=\xi} + pN^b\Big|_{x^3=\eta} = h\partial_b p_a + ge_3\partial_b\left(\frac{h^2}{2}\right) + ge_3 h\partial_b\eta$$

$$\begin{aligned}\partial_t \int_\eta^\xi v^b \mathrm{d}y + \partial_a \int_\eta^\xi v^b v^a \mathrm{d}y + ge_3\partial_b\left(\frac{h^2}{2}\right) = -h\partial_b p_a - ge_3 h\partial_b\eta \\ - v^b\left(v_N - u_N\right)\big|_{x^3=\eta} + \tau^{bi}N_i\big|_{x^3=\eta} - ge_b h + \partial_a \int_\eta^\xi \tau^{ab}\mathrm{d}y - \Sigma_b.\end{aligned} \tag{2.22}$$

The equations still support some simplifications. We neglect the transfer of the momentum from the water body to the soil and assume that atmospheric pressure is constant along water surface. Negleting also the stress viscosity, one has

$$\partial_t \int_\eta^\xi v^b \mathrm{d}y + \partial_a \int_\eta^\xi v^b v^a \mathrm{d}y + ge_3 \partial_b \left(\frac{h^2}{2}\right) = -ge_3 h \partial_b \eta - ge_b h + \tau_f^b. \quad (2.23)$$

τ_{f}^b quantify the friction between water body and soil.

We sumarize our results.

Theorem **2.1 (The Saint Venant equations for overland flow)** *We make the assumptions:*
(H1) *the water is an incompressible Navier-Stokes fluid;*
(H2) *the water viscosity of is of $O(\epsilon)$ magnitude;*
(H3) *the atmospheric medium is an inviscid fluid in the vicinity of the soil surface;*
(H4) *the momentum flux at the both interface, water body-atmosphere and water body-soil, are wall of the $O(\epsilon)$ magnitude.*

Then the first order approximation of the Navier-stokes equation, Saint Venant equations, are given by

$$\begin{aligned} &\partial_t h + \partial_a (h v^a) = r - i, \\ &\partial_t h v^b + \partial_a (h v^b v^a) + \partial_b \left(g e_3 \frac{h^2}{2} \right) = -g e_3 h \partial_b \eta - g e_b h + \tau_f^b. \end{aligned} \quad (2.24)$$

3. Sediment transport

The erosion process is a primary factor that influnces the morfphology of landscape, [12], [28], and it is of crucial importance in the soil conservation technics, [25]. There exists several models to quantifay the sediment loss by soil and its movment by water, [1], [4], [7], [10], [13], [29]. One can rougly clssifies them as empirical model and physically based models. The physically-based erosion model that we present here includes a set of equations derived from the balance of mass and some empiric laws.

The model assumes that:
– there exists a bed of rock soil;
– the bed of rock soil is covered by a stratum of soil that interacts with the flow;
– the sediment consists of two phases: suspended particles in water and

deposited layer;
– there exists a mass exchange between suspended phase and deposited phase;
– the sediment in the deposited layer can only move in vertical direction by saltation process.

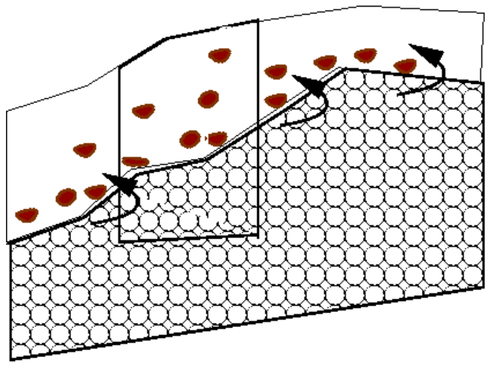


Fig. 4. Configuration of water-suspended sediment- deposited sediment. The soil-water interface is given by $x^3 = \eta(x^1, x^2, t)$ and free surface $x^3 = \xi(x^1, x^2, t)$

Consider a control volume V as in figure (3.). The mass balance equation for sediment in both phases reads as

$$\partial_t \int_b^\xi \rho_s + \partial_a \int_\eta^\xi \rho_s v_s^a \mathrm{d}z = 0. \tag{3.1}$$

By splitting the first integral relatively to deposited layer and water domain, one gets:

$$\partial_t \int_\eta^\xi \rho_s^m \mathrm{d}z + \partial_a \int_\eta^\xi \rho_s^m v_s^a \mathrm{d}z + \partial_t \int_b^\eta \rho_s^d \mathrm{d}z = 0. \tag{3.2}$$

The deposited layer has a constant density that implies

$$\partial_t \int_\eta^\xi \rho_s^m \mathrm{d}z + \partial_a \int_\eta^\xi \rho_s^m v_s^a \mathrm{d}z = -\rho_s^d \partial_t \eta.$$

By introducing the mediate quantities one obtains

$$\partial_t h\overline{\rho_s^m} + \partial_a h q_s^a = -\rho_s^d u_N^{soil}, \tag{3.3}$$

where

$$\begin{aligned}
\overline{\rho_s^m} &= \frac{1}{h}\int_\eta^\xi \rho_s^m \mathrm{d}z, \\
q_s^a &= \frac{1}{h}\int_\eta^\xi \rho_s^m v_s^a \mathrm{d}z, \\
u_N^{soil} &= \partial_t \eta.
\end{aligned} \tag{3.4}$$

On the water - deposited sediment interface, the jump conditions are:

$$\rho_s^m (v_N^s - u_N) = -\rho_s^d u_N. \tag{3.5}$$

4. Overland flow

Concerning the sheet of water flow, the presence of plants on the hill creates a resistance force to the water flow and influences the process of water accumulation on the soil surface. The large diversity of the growing plants on a hill makes very difficult to elaborate an unitary model of the water flow over a soil covered by plants. Here, we present a water mass balance equation that takes into account the presence of certain type of plants.

More precisely, the plants form a dense net of rigid vertical tubes and the water fills the "void" space up to a level not higher than the plant tubes. We obtain a mass balance equation for the system plant water by using a well known technique in the porous media theory i.e. the equations and variables in the equation result from a mediation process on a representative volume, [2].

Let us denote by Ω_f the entire domain of the flow occupied by the fluid and by Ω_p the domanin occupied by plants. For any point $\boldsymbol{x} \in B \subset \mathbb{R}^2$, one defines a representative volume $P_\delta(\boldsymbol{x})$ as

$$P_\delta(x^1, x^2, t) = \{\boldsymbol{y} \in \mathbb{R}^3 || y_a - x^a| < \delta,\ a = \overline{1,2}, \eta(y^1, y^2, t) < y^3 < \xi(y^1, y^2, t)\}. \tag{4.1}$$

Note. One assumes that the bottom surface $x^3 = \eta$ and the upper surface $x^3 = \xi$ are defined in such a way that for any point in the fluid domain they coincide with the soil surface and the free fluid surface, respectively. One defines a plane section of P_δ through a point $\boldsymbol{y}$ of it by

$$S_\delta^a(y^a) = \{\boldsymbol{y}' \in P_\delta(x^1, x^2) \mid y'_a = y_a\}, a = 1 \vee a = 2. \tag{4.2}$$

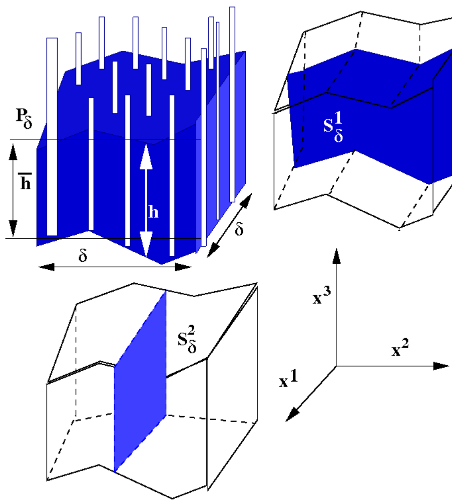


Fig. 5. Reprezentative volume for mediation. In the direction normal to the suport plane one mediates on the entire depth of water and in the plane directions one considers a representative δ width. The depth of water $\widetilde{h}$ in a point is a mediate value of physical h .

Although the water density is considered a constant function, we keep it in the mass balance formulation to put in evidence the physical meaning of the equations. Define the mediate water flux by

$$\rho v^a(\boldsymbol{x},t) = \frac{1}{\mathrm{vol}(\mathrm{P}_\delta(\boldsymbol{x}))}\int_{x^a-\delta}^{x^a+\delta}\int_{S^a_\delta(y^a)\cap\Omega_f}\widetilde{\rho}\widetilde{v}^a(\boldsymbol{y},t)\mathrm{d}\sigma_y \mathrm{d}y^a, \tag{4.3}$$

where $\widetilde{v}^a(\boldsymbol{y},t)$ represents the water velocity field and $\widetilde{\rho}$ mass density of the water.

One defines the mediate water depth by

$$h(x^1,x^2,t) = \frac{\mathrm{vol}(P_\delta(x^1,x^2,t)}{\sigma_\delta}. \tag{4.4}$$

For any P_δ, one writes the mass balance equation

$$\frac{\partial}{\partial t}\int_{P_\delta(\boldsymbol{x},t)\cap\Omega_f}\widetilde{\rho}\mathrm{d}v + \int_{\partial P^f_{f,\delta}(\boldsymbol{x},t)}\widetilde{\rho}\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}\sigma + \tag{4.5}$$

$$+\int_{\partial P^{\mathrm{air}}_{f,\delta}(\boldsymbol{x},t)}\widetilde{\rho}\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}\sigma + \int_{\partial P^{\mathrm{soil}}_{f,\delta}(\boldsymbol{x},t)}\widetilde{\rho}\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}\sigma = 0.$$

The first integral quantifies the water mass contained in the domain P_δ. As the water density is a constant function, one can write

$$\int_{P_\delta(\boldsymbol{x},t)\cap\Omega_f} \widetilde{\rho}\mathrm{d}v = \rho\mathrm{vol}(P_\delta(\boldsymbol{x},t)\cap\Omega_f). \tag{4.6}$$

The second integral accounts for water flux through the boundary of P_δ and the surface $\partial P^f_{f,\delta}$ represents the boundary of P_δ that is in the domain occupied by the fluid. By using the definition of the mediate flux, one can prove that

$$\int_{\partial P^f_{f,\delta}(\boldsymbol{x},t)} \widetilde{\rho\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}v = \partial_a\left(\mathrm{vol}(P_\delta)\rho v^a\right). \tag{4.7}$$

The third and fourth integrals take into account the water supply by rain and water loss by infiltration. The rain contribution can be mediate as

$$\int_{\partial P^{\mathrm{air}}_{f,\delta}(\boldsymbol{x},t)} \widetilde{\rho\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}\sigma = -\rho\sigma_\delta r(\boldsymbol{x},t). \tag{4.8}$$

The infiltration rate is differently mediate. We take into account only the fluid-soil interface

$$\int_{\partial P^{\mathrm{soil}}_{f,\delta}(\boldsymbol{x},t)} \widetilde{\rho\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}\sigma = \rho\sigma^f_\delta i(\boldsymbol{x},t). \tag{4.9}$$

By introducing the definition of the mediate water flux, (4.3), and mediate water depth, (4.4) into (4.7) and (4.6) and by using (4.8) and (4.9), one obtains the water mass balance equation as functional dependence of the mediate quantities

$$\frac{\partial}{\partial t}\left(\frac{\mathrm{vol}(P_\delta(\boldsymbol{x},t)\cap\Omega_f)}{\mathrm{vol}P_\delta(\boldsymbol{x},t)}h(\boldsymbol{x},t)\right)+\partial_a\left(h(\boldsymbol{x},t)v^a\right) = r(\boldsymbol{x},t)-\frac{\sigma^f}{\sigma}i(\boldsymbol{x},t). \tag{4.10}$$

Here are two new quantities that quantify in different manners the plants densities. The first one is the ratio of two volumes

$$n^v(\boldsymbol{x},t) = \frac{\mathrm{vol}(P_\delta(\boldsymbol{x},t)\cap\Omega_f)}{\mathrm{vol}P_\delta(\boldsymbol{x},t)} \tag{4.11}$$

while the second one is the ratio of two areas

$$n^s(\boldsymbol{x},t) = \frac{\sigma^f}{\sigma}. \tag{4.12}$$

Note. *The two ratios play the role of the "porosity" and they have different meannigs and different values. Their values are equal only in the case of constant water depth and perfect cilindrical tubes. For simplicity of formulation, we assume that they are equal and denote by n their common values.*

The final form of the mass balance equation in the presence of the vegetation is

$$\frac{\partial}{\partial t} n(\boldsymbol{x}, t)h(\boldsymbol{x}, t) + \partial_a \left(h(\boldsymbol{x}, t)v^a\right) = r(\boldsymbol{x}, t) - n(\boldsymbol{x}, t)i(\boldsymbol{x}, t). \tag{4.13}$$

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